As the dependence structure is fixed, do more risky assets lead to more risky portfolios?

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Abstract

As the dependence structure (i.e. the copula) among the assets is fixed, one might think that the riskier the assets, the riskier the portfolio. Surprisingly enough, this conjecture turns out to be false even for coherent risk measures and normal returns. We show that two conditions are able to preserve risk ordering under the portfolio: convexity for the risk measure and conditional increasingness for the copula. Eventually, conditional increasingness is checked for the most popular families of copulas used in financial modelling and actuarial sciences.

JEL SUBJECT CLASSIFICATION: G31, G11, C15.

KEY WORDS: *Coherent measures of risk; Portfolio risk models; Copula; Multivariate Stochastic Dominance.*

1 Introduction

Quantitative assessment on the risk involved in a portfolio of financial positions has receiving a growing interest among practitioners and researchers. Mathematically speaking, any risk assessment requires to choose an appropriate risk measure ρ, i.e. a function mapping random variables to real numbers. In their pathbreaking works Delbaen (1998) and Artzner et al. (1999) dwelled what desirable proprieties a risk measure should satisfy. Our aim is to go a step further in this research direction. Let $\mathbf{X} = (X_1, ...X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ be two random vectors collecting the asset returns, such that for every

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fixed $i = 1, .., n$, X_i and Y_i belong to the same business line. The question posed is the following:

> Let $\mathbf{X}$ and $\mathbf{Y}$ have the *same dependence structure*, i.e., a common copula, under which conditions on the risk measure ρ, marginal risk ordering can be preserved under linear combination? i.e.
> if $\rho(X_i) \leq \rho(Y_i)$ for all $i = 1, .., n$, does it imply that
> $\rho\left(\sum_{i=1}^{n} c_i X_i\right) \leq \rho\left(\sum_{i=1}^{n} c_i Y_i\right)$, for all $c_i \in \Re^+$?

One might expect that two portfolios with the same dependence structure, more risky the marginals more risky the portfolios. Surprisingly enough, we show that this conjecture turns out to be false even if ρ is a coherent risk measure and the returns are normally distributed.

The aim of this note is just to deepen this puzzling problem in order to find out guidelines for allocating risks in portfolios. Counter-examples show that risk ordering preservation *is not simply* a question of conditions on the risk measure. But, also restrictions on the dependence structure are compulsory. We enlighten two fundamental gears leading to risk ordering preservation under portfolio: 1) the risk measure should fulfill a *convexity* property (cfr. Axiom 5 Sec. 2.2.1) and 2) the copula should be *conditional increasing* (CI). The former is a relaxed version of sub-additivity which has recently given by Follmer and Scheid (2002), the latter is a mild weaker condition than multivariate totally positive of order 2 (MTP_2).

From the mathematical point of view, CI property seems to be a quite strong condition of positive dependence, but to the best to our knowledge, no weaker condition seems exist. On the other hand, from a practical point of view, this assumption turns out to be not so severe as it may appear at the first sight. CI condition has been checked for the most popular families of copulas. For the elliptical copulas, CI requires the nonnegativeness of correlations. For strict Archimedean copulas, which satisfy Lehmann's positively quadrant dependence, CI imposes only a mild additional condition. Marshall-Olkin copulas are always CI. Exchangeable copulas are CI if further conditions on correlation matrixes are imposed.

The paper is organized as follows. In Section 2 we outline the problem. Sufficient conditions for marginal risk preservation under portfolios are exploited in Section 3. CI condition is tested for the most common families of copulas in Section 4. A conclusion in Section 5 ends the note.

2 Outline of the problem

Let $\mathbf{X} = (X_1, ...X_n)$ and $\mathbf{Y} = (Y_1, ..., Y_n)$ be two random vectors. Let X_i and Y_i , $i = 1, .., n$ be the *excess of return* of the $i - th$ position with reference to a benchmark[1]. For the sake of simplicity, let $E(X_i) = E(Y_i)$, where $E(.)$ is the expectation operator. Let ρ be a measure of risk. Just to fix the ideas, let ρ be the standard deviation or any other more general measure of risk as it will be specified. Now, let us consider the portfolios $\sum_{i=1}^{n} c_i X_i$ and $\sum_{i=1}^{n} c_i Y_i$ with *non-negative weights* $c_i \in \Re^+$. The tackled problem can be worded as follows.

> Let $\mathbf{X}$ and $\mathbf{Y}$ have the *same dependence structure*, if $\rho(X_i) \leq \rho(Y_i)$ for all $i = 1, .., n$, is it true that $\rho\left(\sum_{i=1}^{n} c_i X_i\right) \leq \rho\left(\sum_{i=1}^{n} c_i Y_i\right)$?

At first we have to formalize the meaning of the assumption "$\mathbf{X}$ and $\mathbf{Y}$ must have the *same dependence structure".* This task is achievable by the notion of copula.

2.1 Copulas

An elegant way to understand how a multivariate distribution is influenced by the dependence structure and the marginals is to use the concept of *copula.* This notion has been introduced by Sklar (1959) and studied by Kimeldorf and Sampson (1975) under the name of uniform representation, and by Deheuvels (1978) under the name of uniform representations, (see Schweizer (1991) for a historical overview). In recent years its use has spread out in different fields of insurance and financial modelling (for a rich critical survey on its use in actuarial sciences see Frees and Valdes(1998).

The copula is one of the most useful tool for handling multivariate distributions in the Fréchet class $\Gamma(F_1, ..., F_n)$ of joint n-dimensional distribution functions having $F_1, ..., F_n$ as univariate marginals. Formally, given a distribution function in $\Gamma(F_1, ..., F_n)$, there exists a function $C : [0, 1]^n \to [0, 1]$, such that for all $\mathbf{x} \in \Re^n$,

$$F(\mathbf{x}) = C(F_1(x_1), ..., F_n(x_n))$$

The function C is unique on $\times_{i=1}^{n} Ran(F_i)$, the product of the ranges of F_i, $i = 1, .., n$. Therefore if F is continuous, than C is unique and can be construct as follows:

[1]The benchmark may vary with the position: it could be the liabilities if the $i - th$ position is the return on pension funds, the investment benchmark for traditional asset managers, or just the cash for hedge funds.

$$C(\mathbf{u}) = F\left(F_1^{-1}(u_1), ..., F_n^{-1}(u_n)\right), \qquad \mathbf{u} \in [0,1]^n.$$

Otherwise, C can be extended to $[0,1]^n$ in such a way that it is a distribution function with uniform marginals. Any such extension is called copula of F. Most of the multivariate dependence structure properties of F are in the copula, which does not depend on the marginals, and it is often easier to handle than the original F. Among the numerous proprieties of the copula, we draw the attention to the following.

Proposition 2.1 *The multivariate distributions of two random vectors* $\mathbf{X}$ *and* $\mathbf{Y}$ *have the same (set of) copula(s) if and only if for* $i = 1, .., n$, *there exist strictly increasing functions* h_i *such that* $\mathbf{Y}$ *has the same distribution as* $(h_1(X_1), ..., h_n(X_n))$. *That means that copula is invariant under strictly increasing transformations if the marginals.*

Remark 2.2 *It is worthwhile noting that it is always possible to get the marginal* Y_i *from* X_i *by means of the increasing function* $h_i = G_i^{-1}(F_i)$, *where* F_i *and* G_i *are the marginal of* X_i *and* Y_i, *respectively. In fact*

$G_i(y) = \Pr(Y_i \leq y) = \Pr(h_i(X_i) \leq y) = \Pr\left(X_i \leq h_i^{-1}(y)\right) = F_i\left(h_i^{-1}(y)\right)$ so that $G_i = F_i\left(h_i^{-1}\right)$, then $h_i = G_i^{-1}(F_i)$. So, $Y_i = h_i(X_i)$ if and only if $h_i = G_i^{-1}(F_i)$.□

The relevance of above proprieties stick out as soon as we focus on the distribution of the portfolio $\sum_{i=1}^n c_i X_i,$. Suppose that each asset X_i moves down in risk, so that the marginal return of the $i - th$ business line shifts from F_i to G_i, for $i = 1, .., n$. No change in dependence structure occurs. Formally, the vector $\mathbf{X} = (X_1, ...X_n)$ shifts to $\mathbf{Y} = (h_1(X_1), ..., h_n(X_n)) = (Y_1, ..., Y_n)$ leaving unaltered the copula.

2.2 Measures of risk

In this section we present the essential ideas coming from the seminal work of Artzner et al. (1999) (a further generalization can be found in Delbaen (1998)) leading to the definition of "coherent risk measures". Fix a probability space $(\Omega, \mathcal{F}, P)$ and denote by $L^0(\Omega, \mathcal{F}, P)$ the set of almost surely finite random variables on that space. Financial risks are represented by a *convex cone* $M \subseteq L^0(\Omega, \mathcal{F}, P)$ of random variables. Recall that a M is a convex cone if $X_1 \in M$ and $X_2 \in M$ implies that $X_1 + X_2 \in M X_1 \in M$ and $\lambda X_1 \in M$ for every $\lambda > 0$.

Definition 2.3 *(Risk Measure). Given some convex cone* M *of random variables, any mapping* $\rho : M \to \Re$ *is called a* ***risk measure.***

2.2.1 Axioms

Following Artzner et al. (1999) we enumerate a set of desirable axioms that a risk measure should enjoy. Moreover, in order to simplify the presentation we assume that the risk capital $\rho(X)$ earns no interest.

Axiom 1 (Monotonicity). For all $X_1 \in M$ and $X_2 \in M$, such that $X_1 \leq X_2$ a.s. a monotone risk satisfies $\rho(X_1) \leq \rho(X_2)$.

Axiom 2 (Sub-additivity). For all $X_1 \in M$ and $X_2 \in M$, a subadditive risk measure satisfies $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Axiom 3 (Positive homogeneity). For all $X \in M$ and $\lambda \geq 0$, a positive homogeneous risk measure satisfies $\rho(\lambda X) = \lambda\rho(X)$.

Axiom 4 (Translation Invariance). For all $X \in M$ and $a \in \Re$, a translation-invariant risk measure satisfies $\rho(a + X) = \rho(X) - a$

Axiom 1 says that if a position X_1 is always worth more than X_2, then X_1 cannot be riskier than X_2. Following Artzner et al. (1999, page 209) the rationale behind Axiom 2 can be summarized by the statement "a merger does not create extra risk". Sub-additivity reflects the idea that risk can be reduced by diversification, a well-grounded principle in finance and economics. The lacking of sub-additivity might be an incentive to split up a large portfolio into two smaller ones. That is against to above mentioned statement. Axiom 3 is a limit case of sub-additivity, representing what happens when there is no diversification effect. With reference to Axiom 3, positive homogeneity imposes that the risk of a position increases in a linear way with the size of the position. However, in many situations, it might be not so. For example, an additional liquidity risk may arise if a position is multiplied by a large factor. Follmer and Scheid (2002)) relaxed the conditions of positive homogeneity and sub-additivity and require, instead of Axiom 2 and Axiom 3, the weaker

Axiom 5 (Convexity). For all $X_1 \in M$ and $X_2 \in M$, a risk measure satisfies convexity property if $\rho(\lambda X_1 + (1 - \lambda) X_2) \leq \lambda\rho(X_1) + (1 - \lambda)\rho(X_2)$, for all $\lambda \in [0, 1]$.

Convexity means that diversification does not increase the risk, i.e., the risk of the diversified position $\lambda X_1 + (1 - \lambda) X_2$ is less or equal to the weighted average of the individual risks. In conclusion Axiom 5 captures under looser conditions the fundamental feature of the risk hedging.

Definition 2.4 ***(Coherent Risk Measure)*** *Given a risk measure ρ whose domain includes the convex cone M. ρ is called* ***coherent*** *if it satisfies Axioms 1, 2, 3, and 4.*

Definition 2.5 *(**Convex Risk Measure**) Given a risk measure ρ whose domain includes the convex cone M. ρ is called* **convex** *if it satisfies Axioms 1, 4, and 5.*

Numerous different measures of risk in financial literature exist. In the following, we will handle with some of the most popular. A short list of them is given. Define for fixed $\alpha \in (0,1)$ the α-quantile q_α of ξ by

$$q_\alpha(Z) \quad = \quad \inf\{z \in \mathbb{R} : \ \mathrm{P}[Z \leq z] \geq \alpha\}\,, \tag{2.1}$$

and let $[x]^- = -\min\{x, 0\}$.

Definition 2.6 *Let ξ be a real-valued random variable. Then*

- its *Variance* is

$$\rho_1(\xi) \quad = \quad \mathrm{E}(\xi - \mathrm{E}\,(\xi))^2\,,$$

- its *Standard deviation* is

$$\rho_2(\xi) \quad = \quad \sqrt{\mathrm{E}(\xi - \mathrm{E}\,(\xi))^2}\,,$$

- its *Standard semi-deviation* is

$$\rho_3(\xi) \quad = \quad \sqrt{\mathrm{E}\{[\xi - \mathrm{E}\,(\xi)]^-\}^2}\,,$$

- its *Left-sided Moment of $p-th$ order* $1 \leq p < \infty$ is

$$\rho_4(\xi) \quad = \quad \mathrm{E}\{[\xi - \mathrm{E}\,(\xi)]^-\}^p\,,$$

  if $p = 2$ it coincides with the semi-variance.

- its *Standard left-sided Moment of $p-th$ order* $1 \leq p < \infty$ is

$$\rho_5(\xi) \quad = \quad \sqrt[p]{\mathrm{E}\left[\{[\xi - \mathrm{E}\,(\xi)]^-\}^p\right]}\,,$$

  if $p = 2$ it coincides with the Standard semi-deviation.

- its *Value-at-Risk (VaR)* at level $\alpha \in (0,1)$ is

$$\rho_6(\xi) \quad = \quad \mathrm{E}\,(\xi) - q_\alpha(\xi)\,,$$

- its *Conditional Value-at-Risk (CVaR)* at level $\alpha \in (0,1)$ is

$$\rho_7(\xi) = \ = \mathrm{E}[\xi] - \mathrm{E}[\xi \,|\, \xi \leq q_\alpha\,(\xi)].$$

2.2.2 Properties of the most popular risk measures

Proposition 2.7 *Let $u : \Re \rightarrow \Re$. The risk measure $\rho(\xi) = \mathrm{E}u(\xi)$ satisfies Axiom 5 (Convexity) if and only if u is a convex function.*

Proof. Sufficient condition. By convexity of u

$u(\lambda x + (1-\lambda) y) \leq \lambda u(x) + (1-\lambda) u(y)$ for any $\lambda \in [0,1]$.

taking the expectation on both sides, we see that

$Eu(\lambda x + (1-\lambda) y) \leq E(\lambda u(x) + (1-\lambda) u(y)) = \lambda Eu(X) + (1-\lambda) Eu(Y)$

then the desired Axiom 5 (Convexity) is proved. The necessary condition can be proved by a backward reasoning.

Remark 2.8 *Wang (1996) stated, but not proved, that a concave distortion function g is sufficient for sub-additivity of the risk measure $r(\xi) = \int_0^{+\infty} g(S(x))\,dx$, where S is the decumulative distribution of $-\xi$. Wirch and Hardy (2002, Theorem 2.2) proved that such a condition is necessary and sufficient for sub-additivity. Proposition 2.7 provides a more general proof (since ξ is not to be a nonnegative variable). In fact, we can prove Wirch and Hardy (2002, Theorem 2.2) as a special case of Proposition 2.7. Denoted by F the distribution function of ξ, following relations hold*

$$r(\xi) = \int_0^{+\infty} g(S(x))\,dx = \int_0^{+\infty} g(F(-x))\,dx = \int_{-\infty}^{0} g(F(x))\,dx = \int_{-\infty}^{0} -g(x)\,dF(x)$$

imposing $u = -g$,

$$r(\xi) = \int_{-\infty}^{0} u(x)\,dF(x)$$

so convexity of u is equivalent to concavity of g. In conclusion , the risk measure based on the downside-risk is coherent if and only if u is convex or analogously, the distortion function g is concave.

Proposition 2.9 *Standard moments of even order, as well as Standard left-sided moments satisfy Axiom 5 (Convexity).*

PROOF See Appendix 1.

Remark 2.10 *In conclusion, stand-alone the case of Value-at-Risk, all above ρ enjoy Axiom 5. (Convexity).*

(i) *Variance* enjoys Axiom 5 (Convexity), but falls short in sub-additivity, monotonicity, positive homogeneity, translation. Therefore it is neither a convex nor a coherent risk measure.

(ii) *Standard deviation* satisfies Axiom 5 (Convexity). Moreover, it is positive homogeneous, but it falls short in monotonicity and translation. It is neither a convex nor a coherent risk measure.

(iii) *Standard semideviation* corresponds to the next case, with $p = 2$.

(iv) *Standard left-sided moments* are coherent measures of risk (see Fischer (2002)). Therefore, they are also convex risk measures.

(v) *VaR* fails in convexity and sub-additivity. If fulfills monotony, positive homogeneity and translation. Therefore it is neither a coherent nor a convex risk measure.

(vi) *CVaR* is a coherent risk measure if we restrict ourselves to a convex cone of continuous random variables. A generalized definition coherent for every random variable is given by Acerbi and Tasche (2001). Clearly, it is also convex. Note that CVaR coincides with the so-called stop-loss measure used in Actuarial Sciences. Moreover, if $q_\alpha(\xi) = median$ then CVaR is the left-sided moment of order $p = 1$.

2.3 Lacking in risk ordering for both coherent and convex risk measures

One might think that the phenomenon of lacking in risk ordering under portfolio be related with the fact that the risk measures be or not to be coherent and/or convex. Following counter-examples prove the groundless of this conjecture. Let[2]

$$(X_1, X_2) = N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\right)$$
$$(Y_1, Y_2) = N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}\right)$$

It is easy to prove that $C_X = C_y = C^-$, where C^- is the lower Frechét bound. It turns out that $X_1 + X_2 = N\left(0, 1\right)$ and $Y_1 + Y_2 = N\left(0, 0\right)$. Let us see whether or not marginal risk ordering is preserved.

[2]The seminal idea of this counter-example stems from a case studied by Scarsini (1998) to prove non-preservation of marginal convex ordering into multivariate convex ordering.

Non-coherent risk measures: failure in risk ordering preservation:

- Variance:

even if $var(X_i) < var(Y_i)$ for $i = 1, 2$ it comes out that $var(X_1 + X_2) > var(Y_1 + Y_2)$.

- Standard deviation:

$$\text{even if } st.dev.(X_i) < st.dev.(Y_i) \text{ for } i = 1, 2 \text{ it comes out that}$$
$$st.dev.(X_1 + X_2) > st.dev.(Y_1 + Y_2).$$

- Value-at-Risk at level $\alpha = 0.05$. Since variables involved are normal, it turns out that $VaR(\xi) = 1.65\sqrt{var(\xi)}$. In conclusion,

$$\text{even if } VaR(X_i) < VaR(Y_i) \text{ for } i = 1, 2 \text{ it comes out that}$$
$$VaR(X_1 + X_2) > VaR(Y_1 + Y_2).$$

Coherent Risk Measures: failure in risk ordering preservation.

- Standard semi-deviation: Due to the fact that normal distribution is symmetrical, $\rho(\xi) = (2^{-0.5})\sqrt{var(\xi)}$. So,

$$\text{even if } st.semi - deviation(X_i) < st.semi - deviation(Y_i) \text{ for } i = 1, 2, \text{ but}$$
$$st.semi - deviation(X_1 + X_2) > st.semi - deviation(Y_1 + Y_2).$$

Remark 2.11 *It is worthwhile noting that above example cuts short in what direction we have to drive our research. Since not even normality in distributions and coherence in risk measures are able to guarantee marginal ordering preservation into portfolio, then we are forced to look for additional conditions on the (common) copula.*

3 Convexity in risk measure and Copulas

Let $u : \Re \to \Re$ a *convex* function and $g : \Re \to \Re$ a *strictly increasing* function

$$\rho(\xi) = g[Eu(\xi)] \tag{3.1}$$

Thanks to convexity of u, risk measure (3.1) satisfies Axiom 5 (as it has been proved it is also a necessary condition). Apart VaR, all above mentioned risk measures belong to the family [3] (3.1). Note that VaR admits neither representation (3.1) nor Axiom 5. We will show that is just the *convexity* of u a powerful tool for the object.

With reference to the stochastic dependence, we face the problem into two steps: firstly, we examine the stochastic independence, secondly, the stochastic dependence.

3.1 Stochastic Independence: C=C^0

If component are stochastically independent, copula assumes the simple multiplicative form

$$C^0(\mathbf{u}) = u_1......u_n\ , \qquad \mathbf{u} \in [0,1]^n\,.$$

Under independence, convexity of u drives to the desired risk ordering preservation.

Theorem 3.1 *Let* $\mathbf{X} = (X_1, ...X_n)$ *and* $\mathbf{Y} = (Y_1, ..., Y_n)$ *be vectors with independent components. Let* ρ *admit representation (3.1). Then, the marginal risk ordering is preserved under portfolios.*

Proof. Shaked and Shanthikumar (1994, Theorem 5.A.6) proved that multivariate convex ordering is preserved under independence. Since linear combination is a special convex function, ordering is preserved as well. Therefore it holds the following. If u is a convex function then

$Eu(X_i) \leq Eu(Y_i)$ for all $i = 1,..,n$, implies that $Eu\left(\sum_{i=1}^n c_i X_i\right) \leq Eu\left(\sum_{i=1}^n c_i Y_i\right)$, for all $c_i \in \Re^+$.

Since g is strictly increasing, it preserves the ordering. Since $\rho(\xi) = g\left[Eu(\xi)\right]$, we get

$\rho(X_i) \leq \rho(Y_i)$ for all $i = 1,..,n$, implies that $\rho\left(\sum_{i=1}^n c_i X_i\right) \leq \rho\left(\sum_{i=1}^n c_i Y_i\right)$, for all $c_i \in \Re^+$. as desired.□

Remark 3.2 *VaR fails of preserving risk ordering under linear combinations, even if assets are stochastically independent. A counter-example cuts short any doubts.*

[3]Note that for the one-sided moments $g(t) = \sqrt[p]{t}$ and $f(t) = \left\{|t - E(t)|^-\right\}^p$, $p \geq 2$. With reference to the other risk measures $g(t) = t$ and $f(t) = (t - E(t))^2$ for the variance and $f(t) = t - [t \mid t \leq q_\alpha(t)]$.

Example 3.3 *Let fix confidence level* $\alpha = 0.05$ *and* $C = C^0$ *be fixed. Let consider the two i.i.d.(identical independent distributed) options*

$$X_{1,2} = \begin{cases} -1 & p = 0.04 \\ 0 & p = 0.90 \\ 0.666666 & p = 0.06 \end{cases}$$

where $E(X_{1,2}) = 0$, $VaR(X_{1,2}) = 0$ and $VaR(X_1 + X_2) = 1$. Therefore $VaR(X_1) + VaR(X_2) < VaR(X_1 + X_2)$. In conclusion, VaR fails in convexity for $\mathbf{X} = (X_1, X_2)$. Let consider the two i.i.d. digit options

$$Y_{1,2} = \begin{cases} -1 & p = 0.05 \\ 0.05263 & p = 0.95 \end{cases}$$

Again $E(Y_{1,2}) = 0$, but they result riskier than $X_{1,2}$, because $VaR(Y_{1,2}) = 1$. Since $VaR(Y_1 + Y_2) = 0.9473$, so $VaR(Y_1) + VaR(Y_2) > VaR(Y_1 + Y_2)$, so diversification produces a favorable effect. In conclusion, even though

$$VaR(X_i) < VaR(Y_i) \text{ for } i = 1, 2 \text{ but } VaR(X_1 + X_2) > VaR(Y_1 + Y_2)$$

Remark 3.4 *Some final remarks. (i). Even in the case of stochastic independence, risk ordering preservation under portfolio requires a "sufficiently fitting out" risk measure. (ii) Sub-additivity and homogeneity properties turn out to be redundant restrictions, because single convexity is enough. (iii) Surprisingly, no monotonicity is required. This result re-evaluate the variance as a traditional risk measure for handling portfolio risk.*

3.2 Stochastic dependence

If we replace, however, the assumption of stochastic independence with the assumption of fixed copula, then the conclusion of Theorem 3.1 is not true anymore. Counterexample 1 confirms this statement. Moreover, it is clear that we cannot expect the risk order of the marginals to lead to the risk order of the portfolios, when the components are negatively dependent. The effect of risk hedging may produces a switching in risk ordering between the portfolios. A condition of positive dependence is needed. An extremely rich literature concerning multivariate analysis, statistics, multivariate stochastic majorization exist on this topic.

A notion of positive dependence which fits well in this context is the so-called *conditional increasingness* (CI) recently proposed by Müller and Scarsini (2001). Implicitly, this definition can already be found in Theorem 2 of Rüschendorf (1981). CI is a weaker condition than *multivariate totally positive of order 2* (MTP_2) investigated by Karlin and Rinott (1980) and stronger than *conditional increasingness in sequence* (CIS).(see the Appendix for definitions). CI property coupled with convexity in risk measure gains the desired ordering preservation.

Theorem 3.5 *Let* $\mathbf{X} = (X_1, ...X_n)$ *and* $\mathbf{Y} = (Y_1, ..., Y_n)$ *be vectors with a common CI copula* C*. Let* ρ *admit representation (3.1). Then, the marginal risk ordering is preserved under portfolios.*

Proof. Müller and Scarsini (2001, Corollary 4.6) proved that linear convex order is preserved under nonnegative linear combination if C is CI. Since g is strictly increasing, the order is preserved as well. Since $\rho(\xi) = g\left[Eu\left(\xi\right)\right]$, we get

$\rho\left(X_i\right) \leq \rho\left(Y_i\right)$ for all $i = 1, .., n$, implies that $\rho\left(\sum_{i=1}^{n} c_i X_i\right) \leq \rho\left(\sum_{i=1}^{n} c_i Y_i\right)$, for all $c_i \in \Re^+$. as desired.□

Checking CI property is an easy task in the bivariate case, in fact a complete characterization of CI copulas exists.

Theorem 3.6 *Let* $n = 2$*. A copula* C *is CI if and only if it is concave in each variable when the other is fixed.*

Proof. It is a immediate consequence of Corollary 5.2.11 in Nelsen (1999).

For example, if C belongs to the Plankett family or Ali-Mikhail-Haq family, C has negative pure second partial derivatives, therefore C is CI. Vice versa, for higher dimensions, a simple characterization does not exist. Anyway, a shortcut is available. Note

$$\mathrm{MTP}_2 \rightarrow CI \rightarrow CIS$$

(Müller and Scarsini (2001, Theorem 3.3). In spite of CI, MTP_2 property has been thoroughly studied in the latter twenty years and many families of copulas have been proved to be MTP_2, consequently it is also CI! That is the case of the absolute-value multinormal variables, the multivariate logistic distributions, the negative multivariate among others. Moreover, methods for generating MTP_2 distributions are achievable in Karlin and Rinott (1980).

Remark 3.7 *Theorem 3.5 does not require the property of monotonicity. One might think that using an "increasing" risk measure may permit to relax the striking conditions on the copula. Unfortunately, that is not true. Scarsini (1998, Lemma 1) proved that increasing linear convex is equivalent to linear convex ordering. Therefore, monotonicity has no influence in risk ordering preservation under portfolios.*

4 Are CI the common copulas used in financial modelling?

From a mathematical point of view, CI condition on copula looks quite restrictive. Unfortunately, to the best of our knowledge no alternatives seem to exist in the literature. But two spontaneous questions may arise:

- how "strong" is the CI assumption in financial modelling? and,
- how to test whether CI be or not an acceptable and even a sensible assumption?

With reference to the former question, we will prove that CI holds under mild technical assumptions for the most common families of copulas. This is a good point. Because efficient algorithms for finding the best fitting data copula are available for these families (see Embrechts et al. (2001) and Bouyé et al. (2000)). With reference to the latter question, Denuit and Scaillet (2002) have recently proposed two types of testing procedures for MTP_2. An empirical investigation over US and Danish insurance claim data has been carried out and MTP_2 dependence proved. It is worthwhile noting that MTP_2 is a sufficient condition for CI. So, at least in the cases studied in the literature, CI assumption seems to be acceptable.

4.1 Elliptical copulas

A popular class of distributions for modelling financial market returns is the family of multivariate elliptical distributions which share many of the tractable proprieties of multivariate normal distributions. Elliptical copulas are simply the copulas of elliptical distributions. Simulation from elliptical distributions is simple, and so is the simulation of elliptical copulas. Furthermore, both rank correlation and tail dependence coefficients can be easily calculated. When relaxing the assumption of normality for assets returns it seems natural to look at this family, which contains such distributions as the multivariate

t and the hyperbolic. An example is provided in Frey at al. (2001), where to modelling credit portfolio losses a t-copula is used. In spite of the normal copula, the t-one possesses tail dependence, i.e., expressing the tendency to generate simultaneous extremes values. Therefore its use is suggested to modelling credit portfolios. We get through the study of elliptical copulas into two steps: 1) at first, we consider the case of elliptical copula with elliptical marginals of the same family, 2) at second, an elliptical copula with no elliptical marginals.

Lemma 4.1 *If the random vector* $\mathbf{X}$ *has a multivariate elliptical distribution with copula* C *and invertible correlation matrix* Σ*, then* C *is CI if and only if* Σ^{-1} *is an M-matrix, i.e. its off-diagonal elements are non-negative.*

Proof. In the case $\mathbf{X}$ is a multivariate normal vector, the proof is a simple consequence of Theorem 2 in Rüschendorf (1981). Extensions to general elliptical variables follow from Kelker (1970).□

Note that the concepts of MTP_2 and CI coincide for vectors with invertible covariance matrix.

Theorem 4.2 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be have a multivariate elliptical distribution, with common copula* C *and invertible correlation matrix* Σ *such that* Σ^{-1} *is an M-matrix. Let* ρ *admit representation (3.1). Then, the marginal risk ordering is preserved under portfolios.*

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be normal. Since Σ^{-1} is an M-matrix, for Lemma 4.1 C is CI. Then, for Theorem 3.5 the desired result follows. Extension to elliptical distributions can be derived by Kelker (1970).□

Now, let C be an elliptical copula. If the marginals X_i, Y_i $i = 1, .., n$ are no longer elliptical, the correlation matrix of $\mathbf{X}$ and $\mathbf{Y}$ is no longer equal to the correlation matrix of the elliptical vector with copula C (as it is well-known correlation depends on both the marginals and the copula see for example Schweizer (1991, eq.(6.5)) and Embrechts and et. (2001) for the consequences of this fact). Therefore to check whether C be or not to be CI, we have to construct the correlation matrix of the multivariate normal with copula C. In the positive case, Theorem 4.2 can be extended to the case of non-elliptical marginals.

4.2 Archimedean copulas

For the Archimedean copulas (for the definition see for example Nelsen (1999)) a general representation theorem holds.

Theorem 4.3 *Let φ be a continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1)=0$, and let $\varphi^{[-1]}$ be the pseudo-inverse of φ. Let $C:[0,1]^2 \to [0,1]$ given by*

$$C(u,v)=\varphi^{[-1]}(\varphi(u)+\varphi(v))$$

Then C is a copula if and only if φ is convex. Copulas of such a form are called *Archimedean copulas.* If $\varphi(0)=\infty$ we say that φ is a strict generator. In this case $\varphi^{[-1]}=\varphi^{-1}$ and $C(u,v)=\varphi^{-1}(\varphi(u)+\varphi(v))$ is said to be a *strict Archimedean copula.*

We look at the construction of one particular multivariate extension of Archimedean 2-copulas.

$$C(\mathbf{u})=\varphi^{[-1]}(\varphi(u_1)+....+\varphi(u_n))$$

Kimberling (1974) proved that if C is a strict Archimedean n-copula for all $n \geq 2$ then φ^{-1} is completely monotone. For almost practical purposes, the class of strict generators φ such that φ^{-1} is completely monotone is rich enough class. The *only* generators suitable for extensions to arbitrary dimensions of Archimedean 2-copulas correspond to copulas which can model only positive dependence. In fact, if φ is a strict generator for an Archimedean copula C, then $C \succ C^0$, i.e. $C(u,v) \geq uv$, for all $u,v \in [0,1]$.

Condition $C \succ C^0$ is Lehmann's positively quadrant dependence (PQD). Unfortunately, PQD is a necessary, but *not* sufficient condition for CI (see Nelsen (1999) Theorem 4.7). For a bivariate strict copula a necessary and sufficient condition for CI follows.

Theorem 4.4 *If φ is a twice differentiable strict generator of a bivariate Archimedean copula $C=C(u,v)$, then C is CI if and only if*

-$\frac{\partial}{\partial u}\log\varphi'(u) \geq \frac{\partial}{\partial u}\log\frac{\partial C(u,v)}{\partial v}$ and -$\frac{\partial}{\partial v}\log\varphi'(v) \geq \frac{\partial}{\partial v}\log\frac{\partial C(u,v)}{\partial u}$ for almost all $u,v \in [0,1]$.

Proof. Let calculate the partial derivatives of C:

$\frac{\partial C(u,v)}{\partial u}=\frac{\varphi'(u)}{\varphi'(v)}\frac{\partial C(u,v)}{\partial v}$, therefore

$\frac{\partial^2 C(u,v)}{\partial^2 u}=\frac{1}{\varphi'(v)}\left\{\varphi''(u)\frac{\partial C(u,v)}{\partial v}+\varphi'(u)\frac{\partial^2 C(u,v)}{\partial u\partial v}\right\}$

Note that for absolutely continuous variables $\frac{\partial^2 C(u,v)}{\partial u\partial v}$ is nothing but the density. Analogously $\frac{\partial^2 C(u,v)}{\partial^2 v}$ can be achieved. Imposing the negativeness of second pure derivatives, the desired result comes out.□

An analogous statement can be easily proved: if φ is a differentiable strict generator of a bivariate Archimedean copula $C = C(u, v)$, then C is CI if and only if $log \left[-\frac{\partial}{\partial t} \log \varphi^{-1}(t) \right]$ is convex on $(0, \infty)$. In conclusion, we can state

Theorem 4.5 *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be have a common strict bivariate Archimedean copula* C*, with differentiable strict generator* φ*, such that* $log \left[-\frac{\partial}{\partial t} \log \varphi^{-1}(t) \right]$ *is convex on* $(0, \infty)$. *Let* ρ *admit representation (3.1). Then, the marginal risk ordering is preserved under portfolios.*

4.3 Marshall-Olkin copulas

For bivariate Marshall-Olkin copula, CI property holds.

Theorem 4.6 *Let* $\mathbf{X} = (X_1, ...X_n)$ *and* $\mathbf{Y} = (Y_1, ..., Y_n)$ *be vectors with a common bivariate Marshall-Olkin copula* C*. Let* ρ *admit representation (3.1). Then, the marginal risk ordering is preserved under portfolios.*

Proof. Since C is concave in each variable, as the other is fixed, due to Theorem 3.6 C is CI. So, the statement is proved.□

In general the huge number of parameters for high-dimensional Marshall-Olkin copulas make them unattractive for high-dimensional risk modelling. However, an example of how an easy-to use parametrized model for modelling loss frequencies can be set up, for which the survival copula of times to first losses is just a Marshall-Olkin copula.

4.4 Exchangeable variables

Fully heterogeneous models are extremely difficult to calibrate reliability and it is quite common in practise to segment large portfolios into small number of fairly homogeneous groups, corresponding to the same business lines or same rating class. The correct way to mathematically formalize this notion of homogeneity is to assume that the vector $\mathbf{X}$ is exchangeable (distributionally invariant under permutation)

$$(X_1, ..., X_n) =^d \left(X_{\pi(1)}, ..., X_{\pi(n)} \right)$$

for any permutation $(\pi(1), ..., \pi(n))$ of $(1, ..., n)$. An example of this methodology applied to a huge portfolio of dependent credit risks, which has been decomposed into small exchangeable portfolios, can be found in Frey and McNeil (2002, Sec. 3.5).

Theorem 4.7 *Let* $\mathbf{X} = (X_1, ...X_n)$ *and* $\mathbf{Y} = (Y_1, ..., Y_n)$ *be exchangeable vectors, with* $E(X_i) = \mu_{\mathbf{X}}$ *and* $Var(X_i) = \sigma^2_{\mathbf{X}}$*,* $cov(X_i, X_j) = r_{\mathbf{X}}\sigma^2_{\mathbf{X}}, .E(Y_i) = \mu_{\mathbf{Y}}$ *and* $Var(Y_i) = \sigma^2_{\mathbf{Y}}$*,* $cov(Y_i, Y_j) = r_{\mathbf{Y}}\sigma^2_{\mathbf{Y}}$*. Let* ρ *admit representation (3.1). Then risk ordering under marginals is preserved under portfolio if and only if*

$$\mu_{\mathbf{X}} = \mu_{\mathbf{Y}}, \sigma^2_{\mathbf{X}} \leq \sigma^2_{\mathbf{Y}} \text{ and } \tfrac{\sigma^2_{\mathbf{X}}}{\sigma^2_{\mathbf{Y}}} \leq \max\left\{\tfrac{1-r_{\mathbf{Y}}}{1-r_{\mathbf{X}}}, \tfrac{1+(n-1)r_{\mathbf{Y}}}{1+(n-1)r_{\mathbf{X}}}\right\} \tag{4.4.1}$$

Proof. This follows immediately from Scarsini (1998, Corollary 3).□

Remark 4.8 *Note that if* **X** *and* **Y** *have a common copula, the requirement imposed by Theorem 4.8 does not grow weaker. In fact, a couple of vectors with a same copula do not necessarily has the same correlation matrix, therefore restrictions on the marginal variances still remain.*

In conclusion, in this case conditions are not on the copula, but on the correlation matrix.

Theorem 4.9 *Let* $\mathbf{X} = (X_1, ...X_n)$ *and* $\mathbf{Y} = (Y_1, ..., Y_n)$ *be exchangeable vectors with the same correlation matrix* $\Sigma = cov(X_i, X_j) = cov(Y_i, Y_j)$ *Let* ρ *admit representation (3.1). Then, the marginal risk ordering is preserved under portfolios.*

Proof. If **X** and **Y** have the same correlation matrix, conditions (4.4.1) are trivially satisfied.□

5 Conclusion

As the copula is fixed, no preservation of marginal risk ordering under portfolios is guaranteed. Additional conditions on both the risk measure and the copula are compulsory. Two properties turned out to be powerfull: convexity for the risk measure and conditional increasingness for the copula. This latter is a weaker condition than multivariate totally positive of order 2 (MTP_2) investigated by Karlin and Rinott (1980). CI has been tested for the common families of copulas.

6 Appendix 1

Proposition 6.1 *Standard Left sided moments satisfy Axiom 5 (Convexity)*

Proof. Since for $\forall a, b \in \Re : -(a+b)^{-} \leq -a^{-} - b^{-}$, for any $x \in SuppX$ and $y \in SuppY$

$$-[\lambda x + (1-\lambda) y]^{-} \leq -\lambda x^{-} - (1-\lambda) y^{-} \quad \text{for any } \lambda \in [0,1]$$

since both hand-sides are non-negative, the order is preserved

$$\left(-[\lambda x + (1-\lambda) y]^{-}\right)^{p} \leq \left(-\lambda x^{-} - (1-\lambda) y^{-}\right)^{p}$$

Taking the expectation on both sides

$$E\left(-[\lambda X + (1-\lambda) Y]^{-}\right)^{p} \leq E\left(-\lambda [X]^{-} - (1-\lambda) [Y]^{-}\right)^{p}$$

then

$$\sqrt[p]{E\left(-[\lambda X + (1-\lambda) Y]^{-}\right)^{p}} \leq \sqrt[p]{E\left(-\lambda [X]^{-} - (1-\lambda) [Y]^{-}\right)^{p}}$$

Then, invoking the Minkowski inequality

$$\sqrt[p]{E\left(-\lambda [X]^{-} - (1-\lambda) [Y]^{-}\right)^{p}} \leq \sqrt[p]{E\left(-\lambda [X]^{-}\right)^{p}} + \sqrt[p]{E\left(-(1-\lambda) [Y]^{-}\right)^{p}}$$

$$= \sqrt[p]{\lambda^{p} E\left(-[X]^{-}\right)^{p}} + \sqrt[p]{(1-\lambda)^{p} E\left(-[Y]^{-}\right)^{p}} =$$

$$= \lambda \sqrt[p]{E\left(-[X]^{-}\right)^{p}} + (1-\lambda) \sqrt[p]{E\left(-[Y]^{-}\right)^{p}}$$

In conclusion

$$\sqrt[p]{E\left(-[\lambda X + (1-\lambda) Y]^{-}\right)^{p}} \leq \lambda \sqrt[p]{E\left(-[X]^{-}\right)^{p}} + (1-\lambda) \sqrt[p]{E\left(-[Y]^{-}\right)^{p}}$$

whence the announced result follows.

Proposition 6.2 *Standard $p-th$ order moments, with even p, satisfy Axiom 5 (Convexity).*

Proof. By Minkowski inequality and any even $p \geq 2$

$$\sqrt[p]{E\left|\lambda X - (1-\lambda) Y\right|^{p}} \leq \sqrt[p]{E\left|\lambda X\right|^{p}} + \sqrt[p]{E\left|(1-\lambda) Y\right|^{p}} = \lambda \sqrt[p]{E\left(X^{p}\right)} + (1-\lambda) \sqrt[p]{E\left(Y^{p}\right)}$$

for any $\lambda \in [0,1]$.

Axiom 5 comes out.

7 Appendix 2

In order to the paper be self-contained, definitions are listed. Following dependence concepts grow weaker in sequence.

Definition 7.1 *The random variables* $X_1, ...X_n$ *(or their joint distribution function) are said to be multivariate totally positive of order 2 (MTP*$_2$*) if its density* f *is* MPT_2*, i.e.* $f(\mathbf{u} \vee \mathbf{v}) f(\mathbf{u} \wedge \mathbf{v}) \geq f(\mathbf{u}) f(\mathbf{v})$*, for all* $\mathbf{u} = (u_1, ..., u_n)$ *and* $\mathbf{v} = (v_1, ..., v_n)$ *where*

$$\mathbf{u} \vee \mathbf{v} = (\max(u_1, v_1), ..., \max(u_n, v_n)) \text{ and } \mathbf{u} \wedge \mathbf{v} = (\min(u_1, v_1), ..., \min(u_n, v_n)).$$

Definition 7.2 *The random variables* $X_1, ...X_n$ *(or their joint distribution function) are said to be conditional increasing (CI) if*

$$X_i \uparrow_{st} (X_j, \quad j \in J) \quad \text{for all } J \subset \{1, ..., n\} \quad \text{and } i \notin J.$$

Definition 7.3 *The random variables* $X_1, ...X_n$ *(or their joint distribution function) are said to be conditional increasing in sequence (CIS) if*

$$X_i \uparrow_{st} (X_1, ..., X_{i-1}), \quad i = 2, ..., n$$

that is, if $E(\phi(X_i) \mid X_1 = x_1, X_2 = x_2, ..., X_{i-1} = x_{i-1})$ is an increasing function of the variables $x_1, x_2, ..., x_{i-1}$ for all increasing functions ϕ for which the expectations are defined, $i = 2, ..., n$.

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