Risk Aversion in Cumulative Prospect Theory

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Abstract. This paper characterizes the conditions for risk aversion in cumulative prospect theory when risk aversion is understood as aversion to mean-preserving spreads. Risk aversion implies convex weighting functions for gains and also for losses but not necessarily a concave utility function. However, by assuming additionally continuous weighting functions an overall concave utility is implied. By investigating the exact relationship between loss aversion and risk aversion a natural index for measuring the degree of loss aversion is derived.

Keywords: cumulative prospect theory, risk aversion, loss aversion.

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1 Introduction

Cumulative prospect theory (CPT) has nowadays become one of the most prominent alternative to expected utility (EU). It is widely used in empirical research and, building upon prospect theory (Kahneman and Tversky 1979) and the work of Starmer and Sugden (1989), various axiomatic characterizations of CPT have been proposed (Luce 1991, Luce and Fishburn 1991, Tversky and Kahneman 1992, Wakker and Tversky 1993, Chateauneuf and Wakker 1999, Zank 2001, Wakker and Zank 2002, Schmidt 2002, Schmidt and Zank 2002). The paradoxes of Allais (1953) and Ellsberg (1961) are resolved under CPT, as well as the coexistence of gambling and insurance (Friedman and Savage 1948). The equity premium puzzle (Mehra and Prescott 1985), the overtime premium puzzle (Dunn 1996), the status quo bias (Samuelson and Zeckhauser 1988), and the endowment effect (Thaler 1980) can all be accommodated under the CPT model.

Another popular alternative to expected utility is the rank-dependent utility (RDU) model of Quiggin (1981, 1982). RDU generalizes EU by introducing a weighting function which transforms cumulative probabilities. There exist various theoretical applications of RDU in the literature, showing that it generates results substantially different from those derived in the expected utility framework. In many cases these results provide a better accommodation of observed behavior.

CPT is more general than RDU by allowing additionally for reference-dependence (utility is defined on deviations from a status quo, i.e., on gains and losses, and not on final wealth positions) and sign-dependence (there exist two separate weighting functions, one for probabilities of gains and one for probabilities of losses, which do not need to coincide). Due to reference-dependence a decision maker in the CPT framework can exhibit loss aversion which means that the utility of a given loss weights more heavily than the utility of a gain of equal size. Empirical studies have consistently confirmed loss aversion as a very important aspect of human choice behavior. Rabin (2000) demonstrated that, in the EU-framework, reasonable degrees of risk aversion for small and moderate stakes imply unreasonably high degrees of risk aversion for large stakes (see also Rabin and Thaler 2001). This inconsistency is also implied under RDU (Neilson 2001). Rabin suggests that the empirically observed degrees of risk aversion for small and moderate stakes are in fact a consequence of loss aversion. Hence, the above mentioned inconsistency can be resolved under CPT. Starmer (2000) reviews several recent theories for decision under risk on their merits as descriptive models of choice, and highlights the advantages of rank- and sign-dependence as they are combined in CPT.

A further important aspect of choice behavior is risk aversion. Under EU the curvature of the utility function captures all information concerning risk attitudes. The classical measures of risk aversion established by Pratt (1964) and Arrow (1964) are based on the assumptions of EU where risk aversion is equivalent to concavity of the utility function. Because the modern decision models capture more general preferences a distinction between different forms of risk aversion has emerged. Nearly all theoretical applications of RDU and EU assume strong risk aversion. An individual exhibits strong risk aversion if she or he always dislikes mean-preserving spreads in risk (cf. Rothschild and Stiglitz 1970). In comparison, weak risk aversion holds if a nondegenerate lottery is disliked to its expected value. Chew, Karni and Safra (1987) have shown that strong risk aversion is satisfied within the RDU framework if and only if the utility function is concave and the weighting function is convex. In contrast, strong risk aversion has not yet been analyzed for CPT. Since CPT is more flexible than RDU by incorporating reference-dependence and two weighting functions, the consequences of risk aversion are not obvious. It is useful to derive the conditions for risk aversion under CPT in order to improve our understanding of this model. Moreover, since most theoretical applications of RDU and EU assume strong risk aversion it is natural, as a benchmark case, to do the same for CPT in order to make the results comparable. Most economists associate a concave utility with risk aversion. While under the EU theory risk aversion is equivalent to a concave utility function, it has become clear that under the RDU concavity of the utility function is achieved only with strong risk aversion (Chew, Karni and Safra 1987, Chateauneuf and Cohen 1994). Weak risk aversion is not sufficient to imply this result. It is therefore important to find out whether this association remains valid under CPT.

Our results confirm that the conditions for strong risk aversion under CPT and RDU coincide if we consider only gains or if we consider only losses, i.e., utility is concave on the domain of gains and also concave on the domain of losses and both weighting functions are convex. In contrast to previous studies we derive these results without any prior assumption of differentiability for the utility function because non-differentiability at the status quo is characteristic for CPT. In our proof differentiability[1] is derived entirely from preference conditions, in particular stochastic dominance and strong risk aversion. Our analysis of strong risk aversion allows for discontinuous weighting functions in agreement with most axiomatizantions of CPT. In empirical studies, discontinuities of the weighting functions are frequently observed at impossibility and certainty. One important result is that, if gains and losses are considered simultaneously, surprisingly utility does not need to be concave over the entire domain; in extreme cases strong risk aversion and convex

[1]Our conditions do not imply differentiability of the utility function over the entire domain. Utility will be differentiable except on a countable set.

utility may coexist under CPT. This result is noteworthy since Chateauneuf and Cohen (1994) tried to derive the coexistence of risk aversion and convex utility under RDU. They have shown that RDU is compatible with convex utility only in the presence of weak risk aversion, while strong risk aversion forces utility to be concave everywhere.

Our results show that there exists a particular relationship between strong risk aversion and loss aversion: strong risk aversion is compatible with loss seeking, while utility is concave if and only if loss aversion holds. It turns out that the relationship between strong risk aversion and loss aversion is characterized by the ratio of the left and right derivative of the utility function at zero. Theoretical arguments have motivated Köbberling and Wakker (2001) to propose this ratio as index of loss aversion. Our results support their proposal since in our framework this index of loss aversion arises naturally and entirely from preference conditions.

In order to shed more light on the consequences of strong risk aversion in the CPT framework we consider two specific variants of the model. In the first variant utility is a linear/exponential function, and in the second case it is a power function. By incorporating strong risk aversion these variants lead to examples where the utility function is non-concave, in particular loss seeking behavior is allowed in a region around the status quo. Considering the power function, utility becomes linear for gains and linear for losses with a possible kink at the status quo. In this extreme case utility can be convex, indicating loss seeking behavior. Under the assumption that the weighting functions are continuous, we show that the utility function must be concave everywhere if strong risk aversion is to hold. This shows that loss aversion is satisfied under this additional condition, hence implied by natural and well-known preference conditions.

The paper is organized as follows. In the next section we present the CPT model and derive our main results. Section 3 considers the consequences of strong risk aversion for the two special variants of CPT mentioned above. All proofs are presented in the Appendix.

2 Cumulative Prospect Theory and Risk Aversion

In this section we recall cumulative prospect theory for decision under risk. It is assumed that a decision maker has a preference relation over lotteries denoted by $\succcurlyeq$. A *lottery* is a finite probability distribution over the set of monetary outcomes (here identified with the set of real numbers, $I\!R$). It is represented by $P := (p_1, x_1; \ldots; p_n, x_n)$ meaning that probability p_j is assigned to outcome x_j, for $j = 1, \ldots, n$. The probabilities p_j are nonnegative and sum to one. With this notation we implicitly assume that outcomes are ranked in decreasing order, i.e., $x_1 \geqslant \cdots \geqslant x_n$. Without loss of generality, we assume that the *status quo* is given by zero. Therefore, we refer to positive outcomes as *gains* and to negative outcomes as *losses*.

Cumulative Prospect Theory (*CPT*) holds if the decision maker evaluates lotteries by the following functional.

$$(p_1, x_1; \ldots; p_n, x_n) \mapsto \sum_{j=1}^{n} \pi_j U(x_j),$$

where U is the utility function and the π_j's are decision weights.

The utility function assigns to each outcome a real value with $U(0) = 0$, and it is strictly increasing and continuous. In general, differentiability of the utility function does not need to hold. In particular, all CPT models in the literature allow for non-differentiability at the status quo in order to capture loss aversion.

The decision weights are generated by probability weighting functions w^+, w^-. These functions map the interval of probabilities $[0, 1]$ into itself, they are strictly increasing and satisfy $w^+(0) = w^-(0) = 0$, and $w^+(1) = w^-(1) = 1$. Discontinuities of the weighting functions are explicitly allowed. For a lottery $(p_1, x_1; \ldots; p_n, x_n)$, the decision weights are defined as follows. There exists some $k \in \{0, \ldots, n\}$ such that $x_1 \geqslant \cdots \geqslant x_k \geqslant 0 > x_{k+1} \geqslant \cdots \geqslant x_n$. Then

$$\pi_j = \begin{cases} w^+(p_1 + \cdots + p_j) - w^+(p_1 + \cdots + p_{j-1}), & \text{if } j \leqslant k, \\ w^-(p_1 + \cdots + p_j) - w^-(p_1 + \cdots + p_{j-1}), & \text{if } j > k. \end{cases}$$

Under CPT utility is a ratio scale, i.e., it is unique up to multiplication by a positive constant, and the weighting functions are uniquely determined.

Several axiomatizations of CPT can be found in the literature. To derive the general functional form often complex conditions are required beyond the standard properties (continuity, weak ordering, stochastic dominance). Luce and Fishburn (1991) use a condition termed compound gamble and joint receipt (see also Luce 1991). Tversky and Kahneman (1992), Wakker and Tversky (1993), Chateauneuf and Wakker (1999) and Schmidt (2002) use sign-dependent comonotonic tradeoff-consistency. The conditions can be less complex if a particular parametric form for utility is desired. Schmidt and Zank (2002) use a condition called independence of common increments to derive a model where utility is linear for losses and linear for gains. Wakker and Zank (2002) use a generalization of constant proportional risk aversion to incorporate losses, and derive CPT with utility as power function. Zank (2001) provides a model where utility is exponential or linear by requiring constant absolute risk aversion for gains and separately for losses. The latter two models are analyzed in more detail in the next section.

Recall that under expected utility, where there is no flexibility in modelling distortions in probabilities, risk aversion is entirely captured by concavity of the utility function.

Since the flow of alternatives to expected utility theory different notions of risk aversion have been proposed and analyzed. The most prominent one, introduced by Rothschild and Stiglitz (1970), defines risk aversion as aversion to mean-preserving spreads which is referred to as *strong risk aversion*. More precisely, an individual exhibits strong risk aversion if for all lotteries $P = (p_1, x_1; \ldots; p_n, x_n)$ and $\delta > 0$ it follows that

$$(p_1, x_1; \ldots; p_i, x_i - \frac{\delta}{p_i}; \ldots; p_j, x_j + \frac{\delta}{p_j}; \ldots; p_n, x_n) \precsim P,$$

whenever $p_i, p_j > 0$. Note that, due our notation, δ must be chosen such that rank-ordering of outcomes is maintained.

Several generalizations of strong risk aversion have been proposed by Chateauneuf, Cohen, and Meilijson (2000), and their implications on various decision models have been studied. Strong risk aversion is a property which is model independent, i.e., it is defined in terms of preferences and not in terms of properties of the utility representation. Moreover, the property is sufficient to derive concave utility in EU and RDU. These facts may explain its popularity. In the next theorem we present the implications of strong risk aversion under CPT. Recall that in the case of RDU (characterized by $w^+ \equiv w^-$) utility has to be concave and the weighting function has to be convex if strong risk aversion is to be satisfied. In the theorem below we do not assume any differentiability of the utility function, nor do we assume continuity of the weighting functions. We rather adopt a different approach and show that these assumptions can be relaxed. In this sense Theorem 1 below also generalizes the existing results for RDU.

Theorem 1 *Suppose that cumulative prospect theory holds. Then the following two statements are equivalent:*

1. *Strong risk aversion holds.*

2. The weighting functions w^+ and w^- are convex and therefore continuous on the half-open interval $[0,1)$, the utility function is concave on the domain of losses and also on the domain of gains. Consequently, the utility function and the weighting functions are differentiable except on a countable set. In particular the left and right derivative of the utility function at each outcome exists, as well as the left and right derivatives of the weighting functions at each p in $(0,1)$. Further, the following relationship is satisfied:

$$\frac{U'(0^-)}{U'(0^+)} \geqslant \sup_{p\in(0,1)} \frac{w^{+\prime}(p^-)}{w^{-\prime}(p^+)}. \tag{1}$$

□

Theorem 1 shows that the utility function does not need to be concave on the entire domain since non-concavity at the status quo is permitted. In the next section two special cases of CPT are analyzed with respect to this issue and an example with convex utility is presented.

The formula obtained in Theorem 1 can be used to derive a measure of loss aversion. Since $\sup\limits_{p\in(0,1)} \frac{w^{+\prime}(p^-)}{w^{-\prime}(p^+)}$ can be less than unity for $w^+ \neq w^-$, in general $U'(0^-)/U'(0^+)$ can be also less than unity. Then, some losses may not "loom larger than the corresponding gains", that is, not for all $x > 0$ we have $|U(-x)| \geqslant U(x)$. Therefore, loss aversion does only hold if $U'(0^-)/U'(0^+) \geqslant 1$. Note that an identical definition of loss aversion was already proposed by Bernatzi and Thaler (1995) and formalized by Köbberling and Wakker (2001). The latter authors denote the ratio $U'(0^-)/U'(0^+)$ as index of loss aversion (say λ) and employ it in order to compare the degree of loss aversion between different individuals. In the present paper the measure of loss aversion arises in a natural way, derived solely from preference conditions.

If we want to obtain a utility function in Theorem 1 which is overall concave then $U'(0^-)/U'(0^+)$ must be larger or equal to 1. Since the converse relationship holds as well we can formulate the following corollary.

Corollary 2 *Assume that CPT holds and that strong risk aversion is satisfied. Then, the utility function is concave if and only if loss aversion holds.* □

Note that in the case of RDU we have $w^+ \equiv w^-$ and, therefore, $\sup\limits_{p\in(0,1)} \frac{w^{+\prime}(p^-)}{w^{-\prime}(p^+)} \geqslant 1$. Consequently, in this case strong risk aversion does always imply loss aversion and a concave utility function on the entire domain.

For CPT, two examples which illustrate the issue of non-concavity of the utility function in a region around the status quo are presented in the following section.

3 Examples of Risk Aversion and Convex Utility

As already noted in the preceding section, the interest of economists in a particular parametric form for utility has lead to simpler axiomatizations of CPT. The first functional form for utility which we analyze here is linear/exponential utility. Zank (2001) provides a characterization of CPT with linear/exponential utility for decision under uncertainty. A function $U : I\!R \to I\!R$ is from the *increasing linear/exponential family* for gains (losses) if one of the following holds for all $x \geqslant 0$ ($x \leqslant 0$):

(i) $U(x) = \alpha x$, with $\alpha > 0$,

(ii) $U(x) = \alpha e^{\gamma x} + \tau$, with $\alpha\gamma > 0$ and $\tau \in I\!R$.

Under CPT utility satisfies $U(0) = 0$. Therefore, in (i) we dropped the location parameter, and in (ii) the only possibility for the location parameter is $\tau = -\alpha$. In the above definition only the functional form of utility is described. Clearly the parameters α, γ can be different for gains (say α^+, γ^+) than for losses (say α^-, γ^-). The following result holds if we assume CPT with linear/exponential utility in Theorem 1.

COROLLARY 3 *Suppose that in Theorem 1 cumulative prospect theory holds with linear/exponential utility. Then formula (1) is equivalent to*

$$\lambda := \left\{ \begin{array}{cc} \alpha^-\gamma^-/\alpha^+\gamma^+ & \textit{if utility is exponential} \\ \alpha^-/\alpha^+ & \textit{if utility is linear} \end{array} \right\} \geqslant \sup_{p\in(0,1)} \frac{w^{+\prime}(p^-)}{w^{-\prime}(p^+)},$$

where λ denotes the index of loss aversion. □

If the utility function in the above corollary is exponential for both, gains and losses, the involved parameters are all negative. We can use this result to design an example which shows that utility is not necessarily concave on the entire domain. Suppose that for some positive α the utility function is defined as

$$U(x) = \begin{cases} -\exp(-x) + 1, & x \geqslant 0, \\ \alpha(\exp(-x) - 1), & x \leqslant 0, \end{cases}$$

and for some positive β the weighting functions are defined as

$$w^+(p) = \begin{cases} \frac{\exp(p)-1}{4(e-1)}, & p \in [0,1), \\ 1, & p = 1, \end{cases}$$

and

$$w^-(p) = \begin{cases} \beta w^+(p), & p \in [0,1), \\ 1, & p = 1. \end{cases}$$

Obviously, CPT holds and the above corollary is satisfied if $-\alpha\beta \geqslant 1$. Hence, strong risk aversion holds. However, utility is not concave for $0 < -\alpha < 1$. Non-concavity of the utility will occur if for example $\alpha = -0.9$ and $\beta = 2$.

The next corollary is focusing on a result of Wakker and Zank (2002), in which utility is a two-sided power function of the following form:

$$U(x) = \begin{cases} \sigma^+ x^\alpha, & \text{with } \sigma^+ > 0, \alpha > 0, \text{ for all } x \geqslant 0, \\ -\sigma^- |x|^\beta, & \text{with } \sigma^- > 0, \beta > 0, \text{ for all } x \leqslant 0. \end{cases}$$

This form for utility under CPT has been proposed by Tversky and Kahneman (1992), and is the most used parametric form in empirical and theoretical applications (many references are given in Wakker and Zank 2002). If we assume CPT with power utility in Theorem 1 then the following result holds.

COROLLARY 4 *Suppose that in Theorem 1 cumulative prospect theory holds with power utility. Then formula (1) is equivalent to*

$$\lambda := \frac{\sigma^-}{\sigma^+} \geqslant \sup_{p \in (0,1)} \frac{w^{+\prime}(p^-)}{w^{-\prime}(p^+)},$$

where λ denotes the index of loss aversion. □

For the parameters α, β in the above corollary it holds that $\alpha = \beta = 1$. This follows from the fact that the right hand side of the inequality (1) is a positive constant that is independent of the magnitude of outcomes, and the derivative $U'(0^-) = 0$ as U is concave for losses. The above result shows that strong risk aversion and convex utility can coexist under CPT. Assume that CPT holds with the following utility function

$$U(x) = \begin{cases} \sigma^+ x, & x \geqslant 0, \\ \sigma^- x, & x \leqslant 0, \end{cases}$$

and the weighting functions

$$w^+(p) = \begin{cases} p/4, & p \in [0, 1), \\ 1, & p = 1, \end{cases}$$

and

$$w^-(p) = \begin{cases} p/2, & p \in [0,1), \\ 1, & p = 1. \end{cases}$$

Then for $\sigma^+ = 1$, $\sigma^- = 0.9$ the conditions in the above corollary are satisfied, and obviously utility is convex, due to the fact that utility is less steep for losses than for gains (i.e. $1 = \sigma^+ > \sigma^- = 0.9$).

The reason why in this example we have loss seeking is that the weighting functions are allowed to be discontinuous at 1. If we require continuity of the weighting functions on $[0,1]$ loss-seeking behavior cannot occur. In that case continuity and convexity of the weighting functions imply that $\sup_{p\in(0,1)}[w^{+\prime}(p^-)/w^{-\prime}(p^+)]$ will be at least 1, and, therefore, utility must be concave on the entire domain. The proof is simple. If w^- is above w^+ then for p close to 1 we have $w^{-\prime}(p^+) \leqslant w^{+\prime}(p^-)$. Similarly, if w^- is below w^+ then for p close to 0 we have $w^{-\prime}(p^+) \leqslant w^{+\prime}(p^-)$. If neither of the previous cases holds, then the two weighting functions must intersect. In the case that they overlap on an interval obviously $w^{+\prime} = w^{-\prime}$ there and therefore $w^{+\prime}(p^-)/w^{-\prime}(p^+) = 1$ for some p. Otherwise, there exists an interval in $[0,1]$ where either w^- is entirely above w^+ or where w^- is entirely below w^+ except at the boundary of the interval. Using similar arguments as above, we can conclude the existence of some $p \in (0,1)$ with $w^{-\prime}(p^+) \leqslant w^{+\prime}(p^-)$.

This analysis is independent of the chosen utility function, and therefore it holds in general. This shows that, while under RDU assuming a continuous or discontinuous weighting function is irrelevant for the shape of the utility function, here the assumption of continuous weighting functions is crucial and forces utility to be concave. We get the following result.

Corollary 5 *Suppose that CPT holds and that strong risk aversion is satisfied. Further,* *assume that the weighting functions are continuous on* $[0,1]$*. Then, loss seeking behavior is excluded, i.e., the utility function is concave.* □

4 Conclusion

In the framework of decision under risk, we have contributed towards a better understanding of cumulative prospect theory. In the framework of decision under uncertainty, where probabilities for events are not given in advance, a corresponding analysis would have to invoke an unambiguous definition of uncertainty aversion. Currently there are different versions available most of which are presupposing a particular decision model. However, a model free definition is desirable to allow for a general analysis of loss aversion. An interesting result is that of Epstein (1999) and later Epstein and Zhang (2001) who define uncertainty aversion model free for interpersonal comparisons of risk attitudes. What is needed in addition is a definition that allows for comparisons of different prospects, in particular one that enables an unambiguous statement about when a prospect is considered more uncertain than a second one. The only definition that we are aware of is that of Grant and Quiggin (2001), who provide an intuitive definition of increasing uncertainty which is in many ways related to the definition of mean-preserving spreads in risk. The implications of that definition should be a topic of further research.

Our goal in this paper has been to show what the effects of the assumption of strong risk aversion are for the most popular decision model that permits an analysis of loss attitudes. It is important to fully understand cumulative prospect theory, a decision model which is appealing and, as it has been shown elsewhere, has an impressive predictive power. This model deviates from expected utility in two aspects. The first one is probability weighting, which has been analyzed in depth in the rank-dependent utility framework. The second aspect is loss aversion, which recently receives increasing attention in economic applications. We show that the traditional association of strong risk aversion with a concave utility remains valid under this model only probability weighting functions are assumed continuous, or alternatively, if loss aversion is assumed.

5 Appendix. Proofs

PROOF OF THEOREM 1: Let us first assume Statement 1 and derive Statement 2. Suppose strong risk aversion holds and as well CPT. In what follows we prove that if the outcomes are gains or the status quo then utility must be concave and the weighting function w^+ convex. Then a similar result is derived for the case that outcomes are losses or the status quo: again utility is concave on this domain and the weighting function w^- is convex. In both cases we cannot rely on results from the literature as our assumptions are weaker: we do not assume differentiability of the utility function, rather we derive a certain degree of differentiability from preference conditions. Further we do not assume continuity of the weighting functions. It turns out that these additional assumptions are not necessary. (It has been shown in Corollary 5 that such additional assumptions can be restrictive under CPT.) The final step in the derivation of Statement 2 is to consider the mixed outcome case.

First we consider the lottery $P := (p_1, x_1; \ldots; p_i, x_i; p_{i+1}, x_{i+1}; \ldots; p_n, x_n)$ with outcomes x_i, x_{i+1} of the same sign. Strong risk aversion implies that

$$(p_1, x_1; \ldots; p_i, x_i - \frac{\delta'}{p_i}; p_{i+1}, x_{i+1} + \frac{\delta'}{p_{i+1}}; \ldots; p_n, x_n) \succcurlyeq P$$

whenever $\delta' \in [0, p_i p_{i+1}(x_i - x_{i+1})/(p_i + p_{i+1})]$. After elimination of common terms, substitution of CPT gives

$$\pi_i[U(x_i) - U(x_i - \frac{\delta'}{p_i})] \leqslant \pi_{i+1}[U(x_{i+1} + \frac{\delta'}{p_{i+1}}) - U(x_{i+1})].$$

In particular, for all $p \in (0, 1/2]$, by choosing $p_i = p_{i+1} = p$, we get

$$\pi_i[U(x_i) - U(x_i - \delta)] \leqslant \pi_{i+1}[U(x_{i+1} + \delta) - U(x_{i+1})],$$

for all $x_i > x_{i+1}$ and all $\delta \in [0, (x_i - x_{i+1})/2]$. This implies for all $x_i > x_{i+1}$ and $\delta \in [0, (x_i - x_{i+1})/2]$ and all $p \in (0, 1/2]$

$$\frac{\pi_i}{\pi_{i+1}} \leqslant \frac{[U(x_{i+1} + \delta) - U(x_{i+1})]}{[U(x_i) - U(x_i - \delta)]}. \tag{2}$$

Observe that the left hand side of this inequality is independent of x_i, x_{i+1}, and δ, and similarly the right hand side is independent of p. These facts will be exploited in what follows.

Step 1: Outcomes in P are gains or they are equal to the status quo.

Suppose that there exists $x_i > x_{i+1} \geqslant 0$ and $\delta \in [0, (x_i - x_{i+1})/2]$ such that $U(x_{i+1} + \delta) - U(x_{i+1}) \leqslant U(x_i) - U(x_i - \delta)$. Note that U is strictly increasing, hence these utility differences are all positive. It then follows from (2) that

$$\frac{\pi_i}{\pi_{i+1}} \leqslant 1.$$

Suppose now that there do not exist $x_i > x_{i+1} \geqslant 0$ and $\delta \in [0, (x_i - x_{i+1})/2]$ such that $U(x_{i+1} + \delta) - U(x_{i+1}) \leqslant U(x_i) - U(x_i - \delta)$. Then for all $x_i > x_{i+1} \geqslant 0$ and all $\delta \in [0, (x_i - x_{i+1})/2]$ it follows that $U(x_{i+1} + \delta) - U(x_{i+1}) > U(x_i) - U(x_i - \delta)$. This implies that U is a strictly concave function on the domain of gains (including the status quo). Hence, U is differentiable everywhere except on a countable subset of $I\!R_+$. Further, left and right derivatives exist at any outcome in $I\!R_{++}$ and the right derivative at the status quo. Moreover, all derivatives are positive. We can therefore choose $x_i > x_{i+1} \geqslant 0$ such that $U'(x_i)$ and $U'(x_{i+1})$ exist. It follows that

$$\begin{aligned}\lim_{\delta\to 0}\frac{U(x_{i+1}+\delta)-U(x_{i+1})}{U(x_i)-U(x_i-\delta)} &= \lim_{\delta\to 0}\frac{[U(x_{i+1}+\delta)-U(x_{i+1})]/\delta}{[U(x_i)-U(x_i-\delta)]/\delta} \\ &= \frac{U'(x_{i+1})}{U'(x_i)}\end{aligned}$$

Further by letting x_{i+1} converge to x_i we get

$$\lim_{x_{i+1}\to x_i}\lim_{\delta\to 0}\frac{U(x_{i+1}+\delta)-U(x_{i+1})}{U(x_i)-U(x_i-\delta)} = \lim_{x_{i+1}\to x_i}\frac{U'(x_{i+1})}{U'(x_i)} = 1.$$

Finally, from formula (2),

$$\frac{\pi_i}{\pi_{i+1}} \leqslant 1$$

can be derived for this case.

Hence, $\pi_i/\pi_{i+1} \leqslant 1$ must hold for any $p \in (0, 1/2]$. Substitution of the weighting function w^+ for the decision weights gives

$$w^+(\sum_{j=1}^{i-1} p_j + p) - w^+(\sum_{j=1}^{i-1} p_j) \leqslant w^+(\sum_{j=1}^{i-1} p_j + 2p) - w^+(\sum_{j=1}^{i-1} p_j + p),$$

or equivalently that the weighting function w^+ is convex.

It can now be shown that the weighting function w^+, which is strictly increasing satisfying $w^+(0) = 0$ and $w^+(1) = 1$, is continuous on $[0, 1)$. Convexity and monotonicity are sufficient for the derivation of this property.

Recall that a monotonic continuous function is differentiable almost everywhere. In addition the weighting function w^+ is convex, and this means w^+ is differentiable except on a countable subset of $[0, 1]$, and that the left and right derivatives at each point in

$(0,1)$ exists as well as the right derivative at 0. Discontinuity of the weighting function at 1 cannot be excluded, hence, the left derivative of the weighting function at 1 may not be defined.

The final part of this step is to show that the utility function is indeed concave. Recall that for all $x_i > x_{i+1} \geqslant 0$ and $\delta \in [0, (x_i - x_{i+1})/2]$ and all $p \in (0, 1/2]$

$$\frac{\pi_i}{\pi_{i+1}} \leqslant \frac{[U(x_{i+1}+\delta) - U(x_{i+1})]}{[U(x_i) - U(x_i - \delta)]}$$

holds. Moreover, the right hand side is independent of p. Hence,

$$\sup_{p \in (0,1)} \frac{\pi_i}{\pi_{i+1}} \leqslant \frac{[U(x_{i+1}+\delta) - U(x_{i+1})]}{[U(x_i) - U(x_i - \delta)]}.$$

Note that $\sup\limits_{p \in (0,1)} \frac{\pi_i}{\pi_{i+1}} = 1$ because w^+ is convex and therefore differentiable at some $p \in (0,1)$. Hence,

$$1 \leqslant \frac{[U(x_{i+1}+\delta) - U(x_{i+1})]}{[U(x_i) - U(x_i - \delta)]}$$

for all $x_i > x_{i+1} \geqslant 0$ and $\delta \in [0, (x_i - x_{i+1})/2]$, implying concavity of U. Consequently, U is differentiable on $I\!R_+$, except on a countable subset. We conclude that the left and right derivatives of the utility function at each point $x > 0$ exist and in particular the right derivative at the status quo is well defined. This completes the proof of Step 1.

Step 2: Outcomes in P are losses or they are equal to the status quo.

The proof is similar to the one in the previous step. The same arguments apply to U on $I\!R_-$, and to w^-, respectively. Therefore, we can conclude that the utility function is concave for losses and that the weighting function w^- is convex. Moreover, w^- is continuous on $[0,1)$ and differentiable except on a countable subset of $(0,1)$. The left and right derivative of w^- exists at each point in $(0,1)$, and also the right derivative of

w^- at 0 is well defined. Similarly to the previous step we can conclude that the utility function U is differentiable on $I\!R_-$, except on a countable set, and that the left and right derivative exists at each $x < 0$. In addition the left derivative of U at 0 is well defined (and in general it may not agree with the right derivative established in step 1 of this proof).

Step 3: Some outcomes in P are gains and some losses.

This step will focus entirely on the derivation of the inequality in Statement 2 of the theorem as the remaining statements have been established. Suppose we have a lottery $P = (p_1, x_1; \ldots; p_k, x_k; p_{k+1}, x_{k+1}; \ldots; p_n, x_n)$, where $x_k > 0 > x_{k+1}$. Then strong risk aversion implies

$$(p_1, x_1; \ldots; p_k, x_k - \frac{\delta}{p_k}; p_{k+1}, x_{k+1} + \frac{\delta}{p_{k+1}}; \ldots; p_n, x_n) \succcurlyeq P.$$

Let now $\delta > 0$ be small enough such that $x_k - \delta/p_k > 0 > x_{k+1} + \delta/p_{k+1}$. Substitution of CPT gives

$$\pi_k[U(x_k) - U(x_k - \frac{\delta}{p_k})] \leqslant \pi_{k+1}[U(x_{k+1} + \frac{\delta}{p_{k+1}}) - U(x_{k+1})].$$

Therefore, for any probabilities p_k, p_{k+1} and any $x_k - \delta/p_k > 0 > x_{k+1} + \delta/p_{k+1}$ the following must be satisfied:

$$\frac{[U(x_k) - U(x_k - \frac{\delta}{p_k})]}{[U(x_{k+1} + \frac{\delta}{p_{k+1}}) - U(x_{k+1})]} \leqslant \frac{[w^-(\sum_{j=1}^{k+1} p_j) - w^-(\sum_{j=1}^{k} p_j)]}{[w^+(\sum_{j=1}^{k} p_j) - w^+(\sum_{j=1}^{k-1} p_j)]}.$$

We can write this inequality as

$$\frac{[U(x_k) - U(x_k - \frac{\delta}{p_k})]/\frac{\delta}{p_k}}{[U(x_{k+1} + \frac{\delta}{p_{k+1}}) - U(x_{k+1})]/\frac{\delta}{p_{k+1}}} \leqslant \frac{[w^-(\sum_{j=1}^{k+1} p_j) - w^-(\sum_{j=1}^{k} p_j)]/p_{k+1}}{[w^+(\sum_{j=1}^{k} p_j) - w^+(\sum_{j=1}^{k-1} p_j)]/p_k}.$$

This inequality needs to be satisfied for all probabilities $p_1, \dots, p_n$, all $x_k > 0 > x_{k+1}$ and any appropriate $\delta > 0$ such that $x_k - \delta/p_k > 0 > x_{k+1} + \delta/p_{k+1}$. Hence,

$$\sup_{\substack{x_k - \delta/p_k > 0 > x_{k+1} + \delta/p_{k+1} \\ \delta > 0}} \frac{[U(x_k) - U(x_k - \frac{\delta}{p_k})]/\frac{\delta}{p_k}}{[U(x_{k+1} + \frac{\delta}{p_{k+1}}) - U(x_{k+1})]/\frac{\delta}{p_{k+1}}} \leqslant \inf_{p_1,\dots,p_n} \frac{[w^-(\sum_{j=1}^{k+1} p_j) - w^-(\sum_{j=1}^{k} p_j)]/p_{k+1}}{[w^+(\sum_{j=1}^{k} p_j) - w^+(\sum_{j=1}^{k-1} p_j)]/p_k}$$

must hold. We now use the concavity of the utility function on the gain domain and on the loss domain, and the convexity of the weighting functions on $[0, 1]$, and derive

$$\frac{U'(0^+)}{U'(0^-)} \leqslant \inf_{p \in (0,1)} \frac{w^{-\prime}(p^+)}{w^{+\prime}(p^-)}.$$

This inequality is equivalent to

$$\frac{U'(0^-)}{U'(0^+)} \geqslant \sup_{p \in (0,1)} \frac{w^{+\prime}(p^-)}{w^{-\prime}(p^+)},$$

which concludes the proof of Statement 2.

Let us now assume that Statement 2 holds. Take any lottery $P = (p_1, x_1; \dots; p_n, x_n)$ and let $1 \leqslant i < j \leqslant n$. Define the lottery

$$P(\delta) := (p_1, x_1; \dots; p_i, x_i - \frac{\delta}{p_i}; \dots; p_j, x_j + \frac{\delta}{p_j}; \dots; p_n, x_n)$$

for $\delta \geqslant 0$ such that the rank-ordering of the outcomes is maintained.

If x_i, x_j are of the same sign, then convexity of the corresponding weighting function and concavity of the utility on the respective domain imply $P(\delta) \succcurlyeq P$.

Assume now that $x_i - \frac{\delta}{p_i} > 0 > x_j + \frac{\delta}{p_j}$. We have to show that in this case $CPT(P) - CPT(P(\delta)) \leqslant 0$, or equivalently, that

$$\frac{\pi_i}{\pi_j} \leqslant \frac{U(x_j + \delta/p_j) - U(x_j)}{U(x_i) - U(x_i - \delta/p_i)}.$$

Note that the decision weight π_i is generated by w^+ and π_j is generated by w^-.

We know that

$$\pi_i/\pi_j \leqslant \sup_{p_i,p_j} [\pi_i/\pi_j]$$

and further by using convexity of the weighting functions

$$\sup_{p_i,p_j} [\pi_i/\pi_j] \leqslant \sup_{p\in(0,1)} \frac{w^{+\prime}(p^-)}{w^{-\prime}(p^+)}.$$

Concavity of the utility function on the gain domain and also on the loss domain implies

$$\frac{U'(0^-)}{U'(0^+)} \leqslant \frac{U(x_j+\delta/p_j)-U(x_j)}{U(x_i)-U(x_i-\delta/p_i)}.$$

Using formula (1) from statement 2 of Theorem 1 we conclude

$$\frac{\pi_i}{\pi_j} \leqslant \frac{U(x_j+\delta/p_j)-U(x_j)}{U(x_i)-U(x_i-\delta/p_i)}.$$

or equivalently $P(\delta) \succcurlyeq P$ for this case.

If δ is such that a sign reversal occurs, i.e., $x_i > 0 > x_i - \frac{\delta}{p_i}$ or $x_j + \frac{\delta}{p_j} > 0 > x_j$, then there exists a $0 < \delta' < \delta$ such that $0 = x_i - \frac{\delta'}{p_i}$ (or $x_j + \frac{\delta'}{p_j} = 0$). Repeated application of the previous steps implies first that $P(\delta') \succcurlyeq P$ and second that $P(\delta) \succcurlyeq P(\delta')$. Hence, by transitivity, we conclude $P(\delta) \succcurlyeq P$.

Summarizing the above cases we conclude that strong risk aversion holds, hence statement 1 of the theorem. This concludes the proof of Theorem 1. □

Proof of Corollary 2: The proof follows immediately from the new definition of loss aversion: $\lambda = U'(0^-)/U'(0^+)$. □

Proof of Corollary 3: The proof follows from substitution of the particular form for the utility function in Theorem 1. □

Proof of Corollary 4: The proof follows from substitution of the particular form for the utility function in Theorem 1. □

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