Abstract

In this paper, we analyze the relationship between the reservation unit prices quoted by a market-maker and his level of inventory. Uncertainties about trading volume and the direction of the trade are considered to show that the relationship between the bid-ask spread and the inventory is, in general not monotonic. The paper completes and extends the work of Eeckhoudt-Roger (1999). *JEL classification : G12, D44.*

Reservation prices, Risk aversion and Inventories : the linear case.

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1 Introduction

Madhavan (2000) divides the literature on market microstructure into four main categories : price formation and price discovery, market structure and design issues, information and disclosure, informational issues arising from the interface of market microstructures. The purpose of this paper falls into the first category. More precisely, we analyse the inventory effect on prices in a general inventory model where market makers face four sources of uncertainty ; the direction of the trade, the trade volumes on each side and the terminal payment of the risky asset.

Biais *et al.* (1998) describe trading mechanisms and market structures which vary from an exchange to an other. Prices posted by market makers (referred to as MM in the following) are more or less constrained, depending on the type of market trading mechanism. These constraints affect the flexibility the MM enjoy in proposing the pair (quantity, price). The more flexible trading mechanism allows the MM to define a one-to-one mapping between quantities and unit prices. The dealership market may be considered as the typical example of such an idealized trading mechanism. In this framework bid and ask reservation prices have nice properties[1] as functions of inventories when MM are assumed to be risk averse, as in the Ho and Stoll (1983) model.

However, on floor markets, MM post linear prices after a quantity has been announced by a broker ; Ho and Stoll (1983) and Biais (1993) analyze this case, considering that the entire quantity is traded with one market-maker. Consequently,

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[1] see for example Eeckhoudt-Roger (1999) or Roger (2000) for an analysis of these properties.

there is no uncertainty on traded volumes. However, as orders can in general be split among different MM, the traded volume is in fact subject to uncertainty for the liquidity supplier. To obtain meaningful results when introducing linear pricing and random volumes, one has to set up an upper bound to the quantity to be traded, limit explicitly defined by the broker in the preceding example or, more generally defined by a Normal Market Size, as on the New York Stock Exchange, the Paris Bourse or the Tokyo Stock Exchange.

In this paper, we analyze the properties of reservation bid and ask prices at two levels ; first, we consider the MM who posts prices, conditional on the direction of the trade, that is to say, bid and ask prices are evaluated separately. It is essentially the Ho and Stoll approach, except for the randomness of volumes[2]. This latter feature implies that neglecting simultaneous orders is less justified. When quantities are deterministic and equal on each side, simultaneous orders do not generate a change in net inventories and then are not very interesting on a theoretical point of view. On the contrary, the randomness of traded volumes imply a change in the net inventory, even if simultaneous orders are considered. We show that in this case, the bid-ask spread is tighter than the one obtained in the case of separate evaluation of the two prices.

At a more general level, uncertainty can also arise on the direction of the trade. It is the case when market makers have to post unit bid and ask prices without knowing if the next trade will be on the buy or on the sell side. The difficulty is there to determine simultaneously the two reservation prices. We specialize the analysis by considering usual utility functions depending only on the expectation and the variance of terminal wealth.

We first calculate the initial bid ask spread and show that it is a linear decreasing function of the probability that an order arrives to the market maker. To take into account the inventory effect, a rule must be defined to precise what is called ”reservation price”; in fact, as the traded volume is random, the *ex post* expected utility (after a trade but before the value of the risky asset is revealed) is not equal to the *ex ante* expected utility.

In this paper we select the initial level of utility as the benchmark. We then illustrate that, depending on the volumes traded, spreads can become equal to 0 in some circumstances, especially when the volume in the preceding trade is low.

We then obtain different results, compared to Shen and Starr (2002) who exhibit a systematic relationship between the spread and the inventory, due to the restrictive assumption they impose on the behavior of the market maker. In fact, in their paper, the market maker is assumed to maximize his expected revenue, taking into account a cost function which is increasing in the volume of trade. risk

[2] This point was recently taken into account by Shen and Starr (2002) with different assumptions.

aversion is then not explicitly taken into account.

2 Linear and non linear reservation prices

We consider a risk averse market maker (referred to as the MM in the following) endowed with an initial wealth W_0 and characterized by a strictly increasing and strictly concave utility function U. He has to provide liquidity to the financial market reduced to a single risky asset, the terminal payment of which being a positive random variable X. X is defined on a probability space $(\Omega, \mathcal{A}, P)$ and is assumed square integrable. Moreover, to avoid repetitions of technical conditions, it is assumed that all the random variables appearing in the paper belong to $L^2(\Omega, \mathcal{A}, P)$.

As we want to focus on the inventory risk, we assume that the MM doesn't bear any operational cost, leading him to post reservation bid and ask prices depending on his wealth and his inventory. To keep the presentation as clear as possible in this introductive section, we consider a net inventory α after the first trade.

Three trading mechanisms are possible; the first one can be interpreted as the case of dealership markets. The MM posts different reservation prices for different quantities. We denote as $p_B(\alpha, \beta)$ and $p_A(\alpha, \beta)$ the bid and ask prices for this context. In floor markets, a linear constraint is introduced leading the MM to post unit prices, denoted as $\pi^*_B(\alpha)$ and $\pi^*_A(\alpha)$; the MM has the obligation to provide liquidity up to a quantity α_M of the risky asset (α_M is defined for one trade). However, $\pi^*_B(\alpha)$ and $\pi^*_A(\alpha)$ are determined separately, as in the Ho and Stoll model, this condition meaning that simultaneous trades are not allowed or that prices are posted knowing the direction of the next trade.

Finally, the more general situation involves the randomness of volumes and direction, allowing simultaneous trades. The reservation prices are denoted as $\pi_B(\alpha)$ and $\pi_A(\alpha)$ in this framework. The following definition summarizes the formulation of the reservation prices. The two events A and B have the following definition :

$$\begin{aligned} A &= \{\text{Buying order from the market}\} \\ B &= \{\text{Selling order from the market}\} \end{aligned}$$

Definition 1 *1) $p_B(\alpha, \beta)$ and $p_A(\alpha, \beta)$ are the solutions of the following equations :*

$$\begin{aligned} U(W_0) &= E_X\left[U(W + (\alpha + \beta) X - p_B(\alpha, \beta))\right] \\ U(W_0) &= E_X\left[U(W - (\beta - \alpha) X + p_A(\alpha, \beta))\right] \end{aligned}$$

2) π^*_B and π^*_A are the solutions of the following equations :

$$U(W_0) = E_{X,V_B}\left[U\left(W_0 + \alpha\left(X - \pi^*_B(0)\right) + V_B\left(X - \pi^*_B(\alpha)\right)\right)\right] \quad (1)$$

$$U(W_0) = E_{X,V_A}\left[U(W_0 + \alpha\left(X - \pi_B^*(0)\right) - V_A\left(X - \pi_A^*(\alpha)\right))\right] \tag{2}$$

3) π_B and π_A are a solution of the following equation :

$$U(W_0) = E\left[U(W_0 + \alpha(X - \pi_B(0)) + \mathbf{1}_B V_B(X - \pi_B(\alpha)) - \mathbf{1}_A V_A(X - \pi_A(\alpha)))\right] \tag{3}$$

where $\mathbf{1}_A$ and $\mathbf{1}_B$ are the indicator functions of the events A and B.

E_X is the expectation operator when X is the only random variable at work, E_{X,V_B} and E_{X,V_A} are the expectations operators when the volumes are random and finally, E is the expectation operator with respect to the five variables X, V_A, V_B, $\mathbf{1}_A$, $\mathbf{1}_B$.

It is important to notice that in the first part of the definition, W can be written as $W_0 - p_B(0, \alpha)$ since the MM always posts reservation prices. Moreover, after the trade, the expected utility is equal to $U(W_0)$. It is not the case in the two other definitions ; in fact, prices π^* and π take into account the randomness of volumes; consequently, the expected utility (with respect to X) is in general different after the trade. In others words, when the volume of the first transaction is υ (and $\alpha = 0$), we get :

$$E_X\left[U(W_0 + \upsilon\left(X - \pi_B^*(0)\right))\right] \neq U(W_0)$$

Eeckhoudt-Roger (1999) have shown that $p_B(\alpha, \beta)$ and $p_A(\alpha, \beta)$ are increasing functions of β, $p_B(\alpha, \beta)$ being concave and $p_A(\alpha, \beta)$ convex. Due to risk aversion, the unit prices are not constant, $\frac{p_B(\alpha,\beta)}{\beta}$ $\left(\frac{p_A(\alpha,\beta)}{\beta}\right)$ being decreasing (increasing) with the quantity β to be traded.

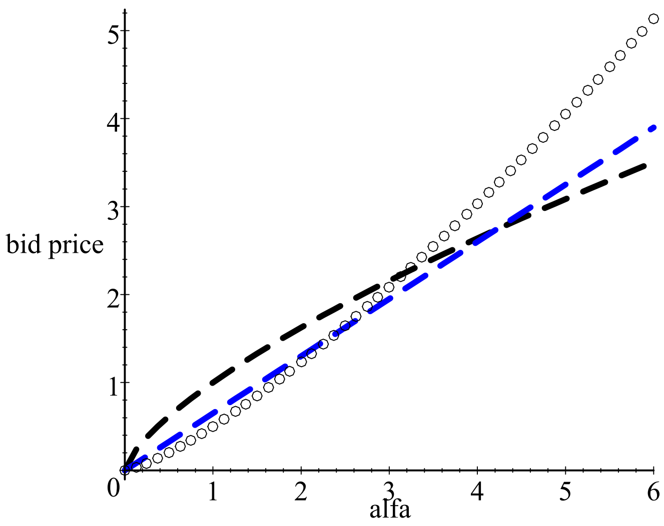
This non-linear pricing rule implies that every Pareto improving exchange can be realized because the seller (buyer) whose reservation ask price for β units is below (above) $p_B(\alpha, \beta)$ ($p_A(\alpha, \beta)$) can trade with the MM..

However, on floor markets, unit prices are posted by market makers and some Pareto-improving trades cannot be realized.

Figure 1 illustrates this point. The price curve of the seller is represented by small circles. It can be seen that orders between 2.5 and 3.5 units would be realized with a non linear pricing rule (the dashed curve) and cannot be executed in the linear world (the dashed line). Consequently, linear pricing ends in a loss of liquidity since some Pareto improving exchanges cannot be managed.

It is also worth to notice that small trades are beneficial to the MM in the linear world when large trades generate a loss in utility. It is then important, for practical purposes, to impose a maximum trade size at the posted prices. It will be denoted as α_M in the following. α_M is often referred to as the NMS (normal market size)

Figure 1: Prices for α units with linear and non linear pricing rules



but in our model, it can be higher and defined by the MM with respect to his initial wealth, for example if his utility function is of the DARA type.

One way for the MM to realize only Pareto-improving exchanges would be to define these prices by the following equalities :

$$\begin{aligned} \phi_B &= \frac{p_B(0, \alpha_M)}{\alpha_M} \\ \phi_A &= \frac{p_A(0, \alpha_M)}{\alpha_M} \end{aligned} \tag{4}$$

However, doing so would lead to an important loss in liquidity because many Pareto improving trades could not be realized. The essential (and obvious) feature of the linear pricing rule is that the unit price doesn't depend on the quantity to be traded, a quantity which is not known in advance by the MM. The "prudent" prices defined in equation 4 cannot be considered as reservation prices because the MM is almost always better off after the trade. This justifies the parts 2 and 3 of the preceding definition. The prices π^* and π, compared to the corresponding definitions of p, can be called *ex ante* reservation prices, the linear constraint being integrated in equations 1 and 2.

3 Properties of π^*

3.1 Reservation prices and inventory

In this section, we analyze the properties of π_B^* and π_A^*; we first consider the inital situation where $\alpha = 0$ and we use the simplified notation $\left(\pi_B^*, \pi_A^*\right)$ instead of $\left(\pi_B^*(0), \pi_A^*(0)\right)$.

The components of the vector (X, V_A, V_B) are assumed to be independent and, for the sake of simplicity, the trading volumes are supposed uniformly distributed on the interval $[0; \alpha_M]$; f is the density of X.

When $\alpha = 0$, equation 1 can be rewritten as :

$$U(W_0) = E_{X,V_B}\left[U\left(W_0 + V_B\left(X - \pi_B^*\right)\right)\right]$$

The independence assumption between V_B and X allows us to write :

$$E_{X,V_B}\left[U\left(W_0 + V\left(X - \pi_B^*\right)\right)\right] = \frac{1}{\alpha_M}\int_0^{\alpha_M}\int_0^{+\infty} U\left(W_0 + v\left(x - \pi_B^*\right)\right) f(x)dx$$

Before analyzing the relationship between the reservation prices and the inventory, we need two preliminary propositions.

Proposition 2 *Let $H(v) = E_X\left[U\left(W_0 + v\left(X - \pi_B^*\right)\right)\right]$; H is a concave function of v and there exists α_0^* such that $H\left(\alpha_0^*\right) = U(W_0)$*

Proof : The concavity of H is obtained immediately by writing :

$$H"(v) = E_X\left[\left(X - \pi_B^*\right)^2 U"\left(W_0 + v\left(X - \pi_B^*\right)\right)\right]$$

which is negative thanks to the concavity of U. The second point is a consequence of the definition of π_B^*. In fact, as we have $H(0) = U(W_0)$ and

$$\frac{1}{\alpha_M}\int_0^{\alpha_M} H(v)dv = U(W_0)$$

the concavity of H implies that H is first increasing then decreasing and the above equality also implies $H(\alpha_M) < U(W_0)$.

The following result shows that the bid price decreases after a first purchase.

Proposition 3 $\forall \alpha > 0, \pi_B^*(\alpha) < \pi_B^*(0)$

Proof : We are going to show that $\frac{1}{\alpha_M}\int_0^{\alpha_M} H(v+\alpha)dv < U(W_0)$; the first term of this inequality is the expected utility of the MM if he buys a random quantity $\alpha + V_B$ at the initial price $\pi_B(0)$. Consequently, if this inequality is verified, he has to lower his second bid price to keep his expected utility unchanged.

$$\begin{aligned}\frac{1}{\alpha_M}\int_0^{\alpha_M} H(v+\alpha)dv &= \frac{1}{\alpha_M}\int_{\alpha}^{\alpha+\alpha_M} H(v)dv \\ &= \frac{1}{\alpha_M}\left[\int_0^{\alpha_M} H(v)dv + \int_{\alpha_M}^{\alpha+\alpha_M} H(v)dv - \int_0^{\alpha} H(v)dv\right] \\ &= U(W_0) + \frac{1}{\alpha_M}\left[\int_{\alpha_M}^{\alpha+\alpha_M} H(v)dv - \int_0^{\alpha} H(v)dv\right]\end{aligned}$$

It is then sufficient to prove that the term between brackets is negative. If $\alpha < \alpha_0^*$, the result is obvious because $H(v) < U(W_0)$ for $v > \alpha_M$ and $H(v) > U(W_0)$ if $v < \alpha$. When $\alpha > \alpha_0^*$, the facts that H is decreasing beyond α_M and $\alpha < \alpha_M$ imply the result, as illustrated in the following example. Consider a MM with a utility function defined by $U(x) = \sqrt{x}$ and an initial endowment $W_0 = 200$. The risky asset follows a two-point distribution $\{10; 15\}$ with equal probabilities and the maximum trade size is $\alpha_M = 15$. The initial bid price is $\pi_B^* = 12.4218$ when the expected value of the risky asset is 12.5.

Figure 2 illustrates the difference between *ex post* expected utility (after a trade $\alpha \in [0; \alpha_M]$) and $U(W_0)$. It appears that for trading volumes lower than 10, the expected utility increases after the trade and decreases sharply beyond this critical level.

Consequently, we get :

$$\int_{\alpha_M}^{\alpha+\alpha_M} H(v)dv - \int_0^{\alpha} H(v)dv < 0$$

The general result concerning the relationship between the bid price and the inventory can now easily be proved.

Proposition 4 *The bid price $\pi_B^*(\alpha)$ is a decreasing function of α.*

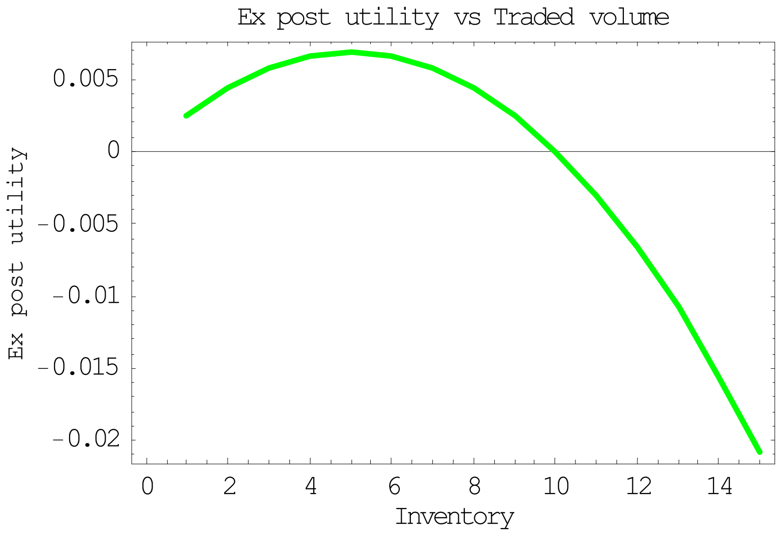
Proof : We can use the preceding relationship :

$$\int_0^{\alpha_M} H(v+\alpha)g(v)dv = U(W_0) + \frac{1}{\alpha_M}\left[\int_{\alpha_M}^{\alpha+\alpha_M} H(v)dv - \int_0^{\alpha} H(v)dv\right]$$

If $\left[\int_{\alpha_M}^{\alpha+\alpha_M} H(v)dv - \int_0^{\alpha} H(v)dv\right]$ is decreasing in α, we are done. However, we get immediately :

$$\frac{\partial}{\partial\alpha}\left[\int_{\alpha_M}^{\alpha+\alpha_M} H(v)dv - \int_0^{\alpha} H(v)dv\right] = H(\alpha+\alpha_M) - H(\alpha)$$

Figure 2: Excess utility after the first trade



which is obviously negative because α is positive and H decreases beyond α_M, that is $H(\alpha + \alpha_M) < H(\alpha_M) < H(\alpha)$.

3.2 The mean-variance case

A usual assumption in microstructure models consists in considering a MM characterized by a utility function depending only on the expectation and the variance of the terminal wealth, that is :

$$E\left[U\left(\widetilde{W}\right)\right] = E\left(\widetilde{W}\right) - a\sigma^2\left(\widetilde{W}\right) \tag{5}$$

where a is a measure of risk aversion. The reservation bid price is then defined by :

$$U\left(W_0\right) = W_0 = W_0 + \frac{1}{\alpha_M}\int_0^{\alpha_M}\left[v\left(\mu_X - \pi_B^*\right) - a\sigma_X^2 v^2\right]dv$$

which is equivalent to :

$$\pi_B^* = \mu_X - \frac{2a\sigma_X^2\alpha_M}{3}$$

Similar calculations lead to $\pi_A^* = \mu_X + \frac{2a\sigma_X^2\alpha_M}{3}$ and the bid-ask spread is $s^* = \frac{4a\sigma_X^2\alpha_M}{3}$.

We can notice a first difference between these prices and those obtained in the non linear world. For a trading volume equal to α, the *ex-post* expected utility is given by :

$$E_X\left[U(W_0+\alpha\left(X-\pi_B^*\right)\right] = W_0+\alpha\left(\mu_X-\pi_B^*\right)-a\sigma_X^2\alpha^2$$
$$= W_0+\frac{2a\alpha\sigma_X^2\alpha_M}{3}-a\sigma_X^2\alpha^2$$

It is greater than $U(W_0)$ as long as $\alpha < \frac{2}{3}\alpha_M$. Moreover, the expected utility is maximum when $\alpha = \frac{\alpha_M}{3}$ and equal to $\frac{a\alpha_M^2\sigma_X^2}{9}$. It means that when a transaction of volume $\alpha < \frac{2\alpha_M}{3}$ is realized, the MM starts the second stage with a utility surplus; he is then able to decrease his ask price and/or increase his bid price; this is obviously not the case when prices are not linear with respect to quantities..

After the first trade $\alpha > 0$, the bid price posted by the MM verifies :

$$\frac{1}{\alpha_M}\int_0^{\alpha_M}\left[\alpha\frac{2a\sigma_X^2\alpha_M}{3}+v\left(\mu_X-\pi_B^*(\alpha)\right)-a\sigma_X^2\left(v+\alpha\right)^2\right]dv=0$$

We then obtain :

$$\alpha\frac{2a\sigma_X^2\alpha_M}{3}+\frac{\alpha_M}{2}\left(\mu_X-\pi_B^*(\alpha)\right)-a\sigma_X^2\frac{(\alpha_M+\alpha)^3-\alpha^3}{3\alpha_M}=0$$

The new bid price is easily obtained with elementary algebraic calculations

$$\pi_B^*(\alpha)=\mu_X-\frac{2a\sigma_X^2}{3\alpha_M}\left(\alpha_M^2+\alpha\alpha_M+3\alpha^2\right)$$

As mentioned in proposition 4, the price $\pi_B^*(\alpha)$ is a decreasing function of α. The difference is equal to :

$$\pi_B^*-\pi_B^*(\alpha)=\frac{2a\sigma_X^2}{3\alpha_M}\left(\alpha\alpha_M+3\alpha^2\right)=a\sigma_X^2\left[\frac{2\alpha}{3}+\frac{2\alpha^2}{\alpha_M}\right] \quad (6)$$

In the non-linear world, the impact of the inventory[3] is valued as :

$$\frac{p_B\left(0,\beta\right)}{\beta}-\frac{p_B\left(\alpha,\beta\right)}{\beta}=a\sigma_X^2\alpha \quad (7)$$

Th difference between the relationships 6 and 7 has a natural economic interpretation. We have :

$$\pi_B^*-\pi_B^*(\alpha)<\frac{p_B\left(0,\beta\right)}{\beta}-\frac{p_B\left(\alpha,\beta\right)}{\beta}\Leftrightarrow\alpha<\frac{\alpha_M}{6}=\frac{2\alpha_M}{3}-\frac{\alpha_M}{2}$$

[3] see Eeckhoudt-Roger (1999), p335

$\frac{2\alpha_M}{3}$ is the level of inventory beyond which the MM losses expected utility and $\frac{\alpha_M}{2}$ is the mean level of each trade (V_B is uniformly distributed on $[0; \alpha_M]$). Consequently, if the MM wants to adjust his expected utility to $U(W_0)$, the changes in prices depend on the expected level of inventory after the next trade. If this level is beyond $\frac{2\alpha_M}{3}$, the bid price decrease is greater than the one observed in the non-linear framework.

The ask price is determined by similar calculations; it verifies :

$$\frac{1}{\alpha_M}\int_0^{\alpha_M}\left[\alpha\frac{2a\sigma_X^2\alpha_M}{3} - w\left(\mu_X - \pi_A^*(\alpha)\right) - a\sigma_X^2(\alpha - w)^2\right]dv = 0$$

which is equivalent to :

$$\frac{\alpha_M}{2}\left(\mu_X - \pi_A^*(\alpha)\right) = \alpha\frac{2a\sigma_X^2\alpha_M}{3} - a\sigma_X^2\frac{\left(\alpha^3 - (\alpha - \alpha_M)^3\right)}{3\alpha_M}$$

and then :

$$\pi_A^*(\alpha) = \mu_X + \frac{2a\sigma_X^2}{3\alpha_M}\left[\alpha_M^2 - 5\alpha_M\alpha + 3\alpha^2\right]$$

The same interpretation can be done here :

$$\pi_A^* - \pi_A^*(\alpha) < \frac{p_A(0,\beta)}{\beta} - \frac{p_A(\alpha,\beta)}{\beta} \Leftrightarrow \alpha < \frac{7\alpha_M}{6} = \frac{2\alpha_M}{3} + \frac{\alpha_M}{2}$$

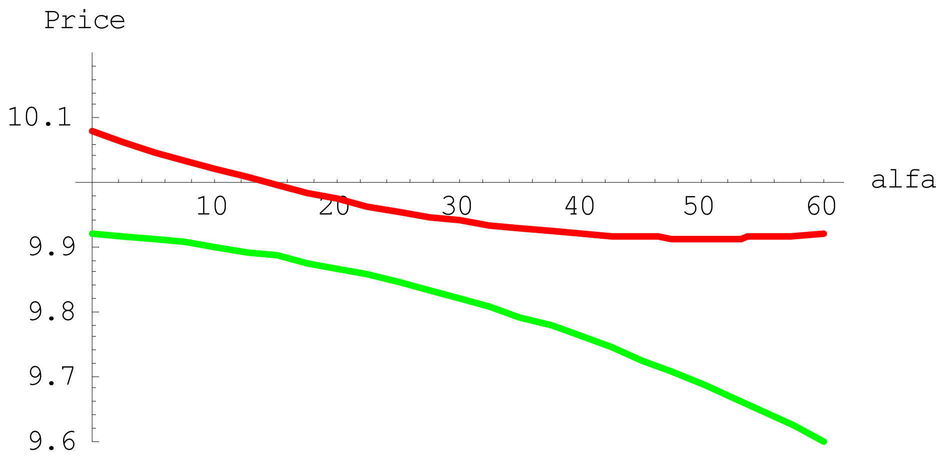
However, it is worth noticing that α being positive, $\frac{p_A(0,\beta)}{\beta} - \frac{p_A(\alpha,\beta)}{\beta} = -a\alpha\sigma_X^2 < 0$. Consequently, if $\alpha < \frac{7\alpha_M}{6}$, the second trade will bring back the market maker to an inventory which generates a surplus of expected utility. He can then lower his ask price more sharply than in the non-linear case. However, with the assumption that α is the inventory after the first trade, the inequality $\alpha < \frac{7\alpha_M}{6}$ is always verified because the maximum trading volume is α_M. It is worth to remark that the ask price doesn't always decrease with inventory, the minimum being obtained for $\alpha = \frac{5\alpha_M}{6}$, a remarkable difference with the non linear case. One more time, this result is not surprising because, as seen before, the maximum *ex post* expected utility is obtained for $\alpha = \frac{\alpha_M}{3} = \frac{5\alpha_M}{6} - \frac{\alpha_M}{2}$, consequently, the MM is brought back to his optimal position after a second trade (on the ask side) equal to the mean volume. Figure 3 represents the evolution of the two prices with respect to α with the following parameters : $\alpha_M = 60$; $a = 0.1$; $\mu_X = 10$; $\sigma_X^2 = 0.1$.

Finally, the bid-ask spread is equal to :

$$s^*(\alpha) = \frac{4a\sigma_X^2}{3\alpha_M}\left(\alpha_M^2 - 2\alpha_M\alpha + 3\alpha^2\right)$$

As illustrated on figure 3, the spread reaches a minimum for $\alpha = \frac{\alpha_M}{3}$, that is the point where the expected utility surplus is the highest.

Figure 3: Bid and ask prices vs inventory



4 The general model

In the preceding section, we essentially considered the buy side, implicitly assuming that bid and ask prices were determined separately; indeed the two prices are indirectly linked by means of the probability distribution of X and the preferences of the MM. However, though we used *ex ante* reservation prices, they were conditional on the direction of the next trade. Going further needs to take into account the uncertainty the MM faces on the direction of the next trade (buy or sell). In this more general approach, there are four sources of risk ; the direction of the trade, the volume of the trade on each side and, finally, the terminal value of the risky asset. This generality was taken into account in the third part of definition 1.

The first source of risk can be represented by the partition of the set of states of nature Ω into four subsets corresponding to the following events.

$B = \{$Buying order and no selling order$\}$

$A = \{$Selling order and no buying order$\}$

$A^c \cap B^c = \{$No order$\}$

$A \cap B = \{$Buying and Selling orders$\}$

In the framework of Ho and Stoll, the first two events have a probability of occurrence equal to $\lambda(1-\lambda)$ meaning that the probability of occurrence of a buying (selling) order is λ; we then have $P(A^c \cap B^c) = (1-\lambda)^2$ and $P(A \cap B) = \lambda^2$. This last event was considered as negligible by these authors ; this was partly justified by their assumption concerning the equal magnitude of buying and selling orders. Under such an hypothesis, simultaneous orders always lead to a wealth increase for the market maker, since the ask price is greater than the bid price, but no change in the net inventory is observed.

As long as a random trading volume is considered, it is difficult to neglect $A \cap B$ because simultaneous purchases and sales of random quantities can move the net inventory of the MM. Consequently, we will now consider general *ex ante* reservation prices as defined by π_A and π_B.

When using the probabilities λ and $1-\lambda$, the condition on expected utility can be rewritten as :

$$\begin{aligned} U(W_0) \;=\; & \lambda(1-\lambda)\left(E_{X,V_B}\left[U(W_0+V_B(X-\pi_B))\right]+E_{X,V_A}\left[U(W_0-V_A(X-\pi_A))\right]\right) \\ & +(1-\lambda)^2 U(W_0)+\lambda^2 E_{X,V_A,V_B}\left[U(W_0+V_B(X-\pi_B)-V_A(X-\pi_A))\right] \end{aligned} \quad (8)$$

One of the main difficulties in this framework comes from the non unicity of the solution. There are infinitely many pairs (π_A, π_B) which verify equation 8. To manage this problem, we use a two-step analysis; first, we determine the bid-ask spread in the general framework and, in a second step, we analyze the deviation generated by this spread from the one obtained when prices are calculated separately.

4.1 The initial bid-ask spread

In this section, we prove that the bid-ask spread is reduced when simultaneous trades are possible, that is to say, when $\lambda > 0$. First of all, we study the properties of the initial spread in this general context. Let us define the function $K(v,w)$ on $[0;\alpha_M] \times [0;\alpha_M]$ as follows :

$$K(v,w) = E_X\left[U(W_0+v(X-\pi_B)-w(X-\pi_A))\right]-U(W_0)$$

$K(v,w)$ is the *ex-post* expected utility surplus when the volumes of trade on each side have been revealed and are equal to v and w.

The initial bid and ask prices then verify :

$$\int_0^{\alpha_M}\int_0^{\alpha_M}\left(\lambda(1-\lambda)\left[K(v,0)+K(0,w)\right]+\lambda^2 K(v,w)\right)dv\,dw = 0$$

which is equivalent to (as $\lambda > 0$) :

$$\int_0^{\alpha_M}\int_0^{\alpha_M}\left((1-\lambda)\left[K(v,0)+K(0,w)\right]+\lambda K(v,w)\right)dv\,dw = 0$$

When considering separately the bid and ask sides (as in the Ho and Stoll model), the following equations are verified simultaneously :

$$\int_0^{\alpha_M} K(v,0)\,dv = 0$$
$$\int_0^{\alpha_M} K(0,w)\,dv = 0$$

Consequently, proving that $\int_0^{\alpha_M}\int_0^{\alpha_M} K(v,w)\,dvdw > 0$ is sufficient to get an improvement of the Ho and Stoll solution by decreasing the ask price and/or increasing the bid price. This is the essential result of this section; before proving it, we need several technical lemmas.

Lemma 5 *K is a concave function on* $[0;\alpha_M]\times[0;\alpha_M]$

Proof : As K depends on two variables, it is concave if the principal minors of the Hessian matrix alternate in signs, the first being negative. In fact, we know that $\frac{\partial^2 K}{\partial v^2} < 0$ because of the concavity of U since :

$$\frac{\partial^2 K}{\partial v^2} = E_X\left[(X-\pi_B)^2\,U"(W_0 + v(X-\pi_B) - w(X-\pi_A))\right] < 0$$

It is then sufficient to prove that the determinant of the Hessian matrix is positive.

$$D = \frac{\partial^2 K}{\partial v^2}\frac{\partial^2 K}{\partial w^2} - \left(\frac{\partial^2 K}{\partial v \partial w}\right)^2$$

In fact, skipping the argument of $U"$, D can be written as :

$$D = E_X\left[(X-\pi_B)^2\,U"(.)\right]E_X\left[(X-\pi_A)^2\,U"(.)\right] - \left(E_X\left[(X-\pi_B)(X-\pi_A)\,U"(.)\right]\right)^2$$

To sign this expression, we use a change of probability by writing :

$$\frac{D}{E_X[U"(.)]^2} = E_X\left[(X-\pi_B)^2\frac{U"(.)}{E_X[U"(.)]}\right]E_X\left[(X-\pi_A)^2\frac{U"(.)}{E_X[U"(.)]}\right] - \left(E_X\left[(X-\pi_B)(X-\pi_A)\frac{U"(.)}{E_X[U"(.)]}\right]\right)^2$$

Let Q the probability equivalent to P_X defined by its Radon-Nikodym derivative $\frac{dQ}{dP_X} = \frac{U"(.)}{E_X[U"(.)]}$. We get :

$$\frac{D}{E_X[U"(.)]^2} = E_Q\left[(X-\pi_B)^2\right]E_Q\left[(X-\pi_A)^2\right] - \left(E_Q\left[(X-\pi_B)(X-\pi_A)\right]\right)^2$$

Denoting now $X_A = X - \pi_A$ and $X_B = X - \pi_B$, we obtain :

$$\frac{D}{E_X\left[U"(.)\right]^2} = \|X_A\|_{L^2}^2 \, \|X_B\|_{L^2}^2 - \langle X_B, X_A\rangle_{L^2}^2$$

where $\|.\|_{L^2}$ and $\langle ., .\rangle_{L^2}$ denote the L^2-norm and the corresponding inner product. The Cauchy-Schwarz inequality then implies that $D > 0$.

Consider now the square $[0; \alpha_M] \times [0; \alpha_M]$; the function $K(v, w)$ is increasing along the diagonal because we have :

$$K(\alpha, \alpha) = E_X\left[U(W_0 + \alpha(\pi_A - \pi_B))\right] - U(W_0)$$

and $\pi_A > \pi_B$ by Jensen's inequality.

Lemma 6 $\forall (v, w) \in [0; \alpha_M] \times [0; \alpha_M]$ *we have the following inequalities :*

$$\begin{aligned} v &> w \Rightarrow K(v, w) > K(v - w, 0) \\ w &> v \Rightarrow K(v, w) > K(0, w - v) \end{aligned}$$

Proof :

$$\pi_A > \pi_B \Rightarrow -w(X - \pi_A) > -w(X - \pi_B)$$

We deduce immediately :

$$K(v, w) > E_X\left[U(W_0 + (v - w)(X - \pi_B))\right] - U(W_0) = K(v - w, 0)$$

The same argument works for the second inequality.

This lemma proves a very intuitive result, that is the function K is increasing along all the lines with slope equal to one. Along these lines, the net position of the MM is equal but the quantity sold during the simultaneous trade is increasing. As the bid-ask spread is positive, the sure wealth of the MM increases along the lines.

We can now easily prove the main result.

Proposition 7 *The expected utility generated by simultaneous orders is positive, that is :*

$$A = \int_0^{\alpha_M} \int_0^{\alpha_M} K(v, w) dv dw > 0$$

Proof :

$$\begin{aligned} A &= \int_0^{\alpha_M} \left[\int_0^{w} K(v, w) dv + \int_w^{\alpha_M} K(v, w) dv\right] dw \\ &= \int_0^{\alpha_M} \left[\int_0^{w} K(v, w) dv + \int_0^{\alpha_M - w} K(v + w, w) dv\right] dw \end{aligned}$$

The first part of lemma 6 implies :

$$A > \int_0^{\alpha_M} \left[\int_0^{w} K(v, w)dv + \int_0^{\alpha_M - w} K(v, 0)dv \right] dw$$

As we know that $\int_0^{\alpha_M} K(v, 0)dv = 0$ and $K\,(v, 0)$ is concave as a function of v (it is in fact the function $H(v)$ of the preceding section), we obtain :

$$\forall w < \alpha_M, \int_0^{\alpha_M - w} K(v, 0)dv > 0$$

Consider now the first term $\int_0^{w} K(v, w)dv$; using lemma 6 leads to :

$$\int_0^{w} K(v, w)dv > \int_0^{w} K(0, w - v)dv = \int_0^{w} K(0, y)dy$$

with the obvious change in variable $y = w - v$. One more time using the concavity of $K\,(0, y)$ as a function of y and the equality $\int_0^{\alpha_M} K(0, y)dy = 0$, we get $\int_0^{w} K(v, w)dv > 0$ which ends the proof.

4.2 A numerical example

To illustrate our proposition, we consider a risky asset taking two values 5 and 10 with equal probabilities. The market maker possesses an initial wealth equal to 100 and $\alpha_M = 10$. We select a logarithmic utility function and the reservation prices, when considered separately, are :

$$\begin{aligned} \pi_B &= 7.293 \\ \pi_A &= 7.707 \end{aligned}$$

The large bid-ask spread comes from the low initial wealth of the MM.

We then obtain :

$$\int_0^{\alpha_M} \int_0^{\alpha_M} K(v, w)dvdw = 1.54$$

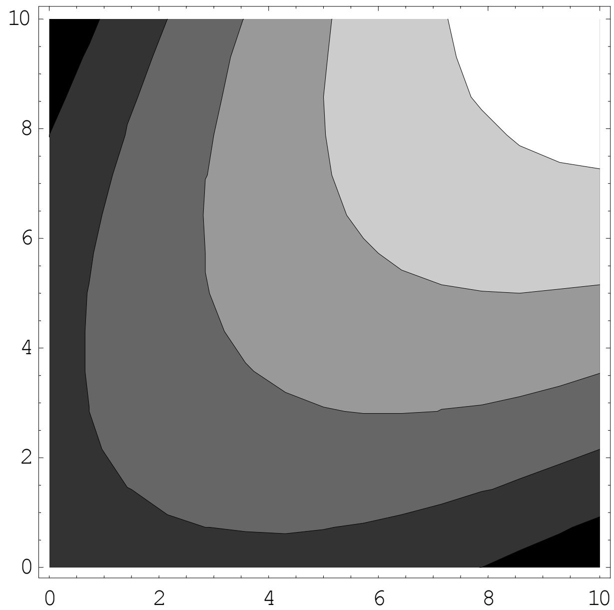
In fact, the utility surplus is equal to $1.54/\alpha_M^2 = 0.0154$.

Figure 4 represents the level curves of K. As mentioned before, the level is lower on the north-west and south-east parts (black color).

The general equation defining the reservation prices is given by :

$$F\,(\pi_A, \pi_B) = \int_0^{\alpha_M} \int_0^{\alpha_M} \left(\lambda\,(1 - \lambda)\,[K\,(v, 0) + K\,(0, w)] + \lambda^2 K\,(v, w) \right) dvdw = 0$$

Figure 4: Level curves of K



Consequently, there is an infinite number of solutions. To obtain a good approximate solution, we first consider the prices estimated separately, that is π_B^* and π_A^* verifying :

$$\begin{aligned}\int_0^{\alpha_M} K(v,0)\,dv &= 0\\ \int_0^{\alpha_M} K(0,w)\,dw &= 0\end{aligned}$$

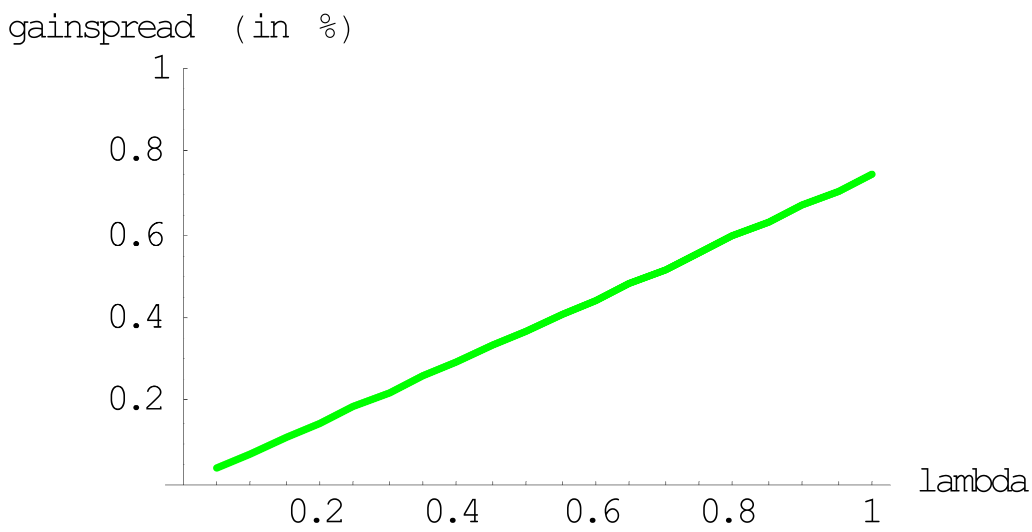
We then define the following function :

$$K^*(v,w) = E_X\left[U\left(W_0 + v\left(X - \pi_B^* - \varepsilon\right) - w\left(X - \pi_A^* + \varepsilon\right)\right)\right]$$

ε is a measure of the improvement of the reservation prices when one takes into account the result of proposition 7. Figure 5 illustrates the reduction of the spread, measured as $\frac{2\varepsilon}{\pi_A^* - \pi_B^*}$ with respect to λ, the probability of arrival of an order (either buying or selling order).

It appears, on the data of our example, that the reduction of the spread is far from negligible, even for low values of λ.

Figure 5: Reduction of the spread (in %) versus λ



However, the spread is initially large because of the unrealistic volatility assumed on X. It must be kept in mind that X is the terminal value of the risky asset but, in our framework, it can be interpreted as the closing price of the stock because the horizon of microstructure models is, in general, very short.

4.3 The analytical solution in the mean-variance world

We consider here the same assumption as in section 4.1, that is the expected utility of the MM only depends on the first two moments of the distribution of future wealth. Keeping the notations of the preceding section, we define :

$$\begin{aligned} K(v,0) &= v\left(\mu_X - \pi_B\right) - av^2\sigma_X^2 \\ K(0,w) &= w\left(\pi_A - \mu_X\right) - aw^2\sigma_X^2 \\ K(v,w) &= v\left(\mu_X - \pi_B\right) + w\left(\pi_A - \mu_X\right) - a\left(v-w\right)^2\sigma_X^2 \end{aligned}$$

where μ_X and σ_X^2 are the expectation and the variance of X. The reservation prices are then a solution of :

$$\frac{1}{\alpha_M^2}\int_0^{\alpha_M}\int_0^{\alpha_M}\left((1-\lambda)\left[K(v,0)+K(0,w)\right]+\lambda K(v,w)\right)dvdw=0$$

The analytical formulation of K allows us to compute explicitly the spread $\pi_A - \pi_B$ and to compare it to the one obtained when the two sides are valued separately.

Proposition 8 *The bid-ask spread $s(\lambda)$ is defined as :*

$$s(\lambda)=\pi_A-\pi_B=a\alpha_M\sigma_X^2\left[\frac{4}{3}-\lambda\right]$$

The proof is reported in the appendix.

We can remark that when λ tends to 0, the spread converges to the spread obtained when the two prices are evaluated separately ; in fact, we know that :

$$\begin{aligned}\pi_A^* &= \mu_X+\frac{2}{3}a\alpha_M\sigma_X^2\\ \pi_B^* &= \mu_X-\frac{2}{3}a\alpha_M\sigma_X^2\end{aligned}$$

Moreover, when $\lambda = 1$, the spread is equal to $\frac{1}{3}a\alpha_M\sigma_X^2$; the randomness of trading volumes generates an uncertainty on the net position of the MM, even if his expected net position after the trade is equal to zero. Obviously, in a world of fixed and equal quantities for the two sides, the spread would be zero to keep the expected utility of the MM unchanged.

To summarize, the spread is an increasing function of risk aversion, of the volatility of the terminal payment of the risky asset and of the maximum quantity the MM is obliged to trade. All these results are quite intuitive. On the contrary, the spread is decreasing in λ and, one more time this is not surprising because λ^2 is the probability that two simultaneous orders arrive to the MM. When this probability is high, the MM can partially hedge his position even if he doesn't control the quantities to be traded on each side.

We can now easily compare the spread we have obtained with the one deduced from Ho and Stoll model when the quantity to be traded is known. Indeed, for the two spreads to be comparable, we have to consider a quantity $Q = \frac{\alpha_M}{2}$, that is the mean of our random volumes V_A and V_B.

Corollary 9 *Let s_{HS} the Ho and Stoll spread for the quantity $\frac{\alpha_M}{2}$; we then have :*

$$\begin{aligned} s(0) &= \frac{4}{3}s_{HS} \\ s\left(\frac{1}{3}\right) &= s_{HS} \end{aligned}$$

Proof :s_{HS} is obtained directly by the following equations (deduced from the utility function (5)) :

$$\begin{aligned} \frac{\alpha_M}{2}\left(\mu_X - \pi_B\right) - a\sigma_X^2\left(\frac{\alpha_M}{2}\right)^2 &= 0 \\ \frac{\alpha_M}{2}\left(\pi_A - \mu_X\right) - a\sigma_X^2\left(\frac{\alpha_M}{2}\right)^2 &= 0 \end{aligned}$$

We get immediately :

$$s_{HS} = a\sigma_X^2\alpha_M$$

which is equal to $\frac{3}{4}s(0)$. It also follows that $s\left(\frac{1}{3}\right) = s_{HS}$.

In fact, introducing random volumes and the possibility of simultaneous trades induce two opposite effects. Random volumes increase the spread by $33.\frac{1}{3}\%$ and simultaneous trades (with probability λ^2) decrease the spread by $\frac{3}{4}\lambda\%$ (with respect to the spread with only random volumes).

5 Spread and inventory in the mean-variance model

Suppose now that the MM has first traded a quantity $\alpha > 0$ at a price π_B. The question is to know if $\pi_B(\alpha)$ is lower than the initial buying price.The two reservation prices in the framework of the preceding section are now formulated by means of a three-variable function $M(\alpha, v, w)$ defined by :

$$M(\alpha, v, w) = E_X\left[U\left(W_0 + \alpha\left(X - \pi_B(0)\right) + v\left(X - \pi_B(\alpha)\right) - w\left(X - \pi_A(\alpha)\right)\right)\right] - U(W_0)$$

The definition of the prices $\pi_A(\alpha)$ and $\pi_B(\alpha)$ are adapted to the new notation in the following way.

Definition 10 *$\pi_B(\alpha)$ and $\pi_A(\alpha)$ are a solution of the following equation:*

$$\int_0^{\alpha_M}\int_0^{\alpha_M}\left(\lambda(1-\lambda)\left[M(\alpha, v, 0) + M(\alpha, 0, w)\right] + \lambda^2 M(\alpha, v, w) + (1-\lambda)^2 M(\alpha, 0, 0)\right)dv\,dw = 0 \quad (9)$$

The most important remark concerns the term $\int_0^{\alpha_M}\int_0^{\alpha_M} M(\alpha, 0, 0)dvdw$; it was equal to zero in the initial situation $\alpha = 0$; however, as it represents the expected utility surplus generated by the first trade, it is no more the case when $\alpha > 0$, depending on the value of α in the interval $[0; \alpha_M]$.

It is easy to prove that $M(\alpha, v, w)$ is concave in (v, w) whatever the inventory α is but $M(\alpha, 0, 0)$ being in general different from 0, we cannot use the same method as in proposition 7.

As there are many pairs $(\pi_B(\alpha), \pi_A(\alpha))$ which verify equation 9, we perform two analyses ; first, keeping the problem in its more general formulation, we determine the bid-ask spread $\pi_A(\alpha) - \pi_B(\alpha)$. Nothing more can be said about the individual prices in this context. In a second step, we consider the prices $\pi_A^*(\alpha)$ and $\pi_B^*(\alpha)$ evaluated in section 3.2 and we solve the general model by defining ε such that the relation (**??**) is verified for $\pi_A^*(\alpha) - \varepsilon$ and $\pi_B^*(\alpha) + \varepsilon$.

Proposition 11 *After a trade of volume α, the spread $s(\alpha)$ is given by :*

$$s(\alpha) = \frac{2a\sigma_X^2}{\lambda\alpha_M}\left[\alpha^2 + \lambda\alpha_M^2\left(\frac{2}{3} - \frac{\lambda}{2}\right) - \alpha\alpha_M\left(\frac{2}{3} - \frac{\lambda}{2}\right)\right]$$

The proof, containing tedious calculations, is given in the appendix.

An "interesting" property of $s(\alpha)$ is the fact that it can be negative. Suppose that the first trade α let the MM with the highest utility surplus, that is $\alpha = \frac{\alpha_M}{3}$. We get :

$$s(\alpha) = \frac{2a\sigma_X^2\alpha_M}{\lambda}\left[\frac{1}{9} + \left(\lambda - \frac{1}{3}\right)\left(\frac{2}{3} - \frac{\lambda}{2}\right)\right]$$

For λ sufficiently close to 0, $s\left(\frac{\alpha_M}{3}\right)$ is negative; It can be easily explained by a careful examination of the definition **??**. When λ is near 0, the most weighted term is the one corresponding to no trade $\left(1 - \lambda\right)^2 M(\alpha, 0, 0)$. However, this term is in fact the surplus generated by the first trade, which is positive if $\alpha = \frac{\alpha_M}{3}$. Consequently, to compensate this surplus, the other terms must be largely negative; it can only be obtained by increasing (decreasing) sharply the bid (ask) price.

Of course, this result is not consistent, first on a practical point of view and, second because of the assumption of independence of variable V_A, V_B, $\mathbf{1}_A$ and $\mathbf{1}_B$. It is clear that if the spread was negative in a world where simultaneous trades are allowed, arbitrage opportunities would arise immediately. To be consistent with our preceding assumptions, we have to consider a minimum spread equal to 0.

The second question is to know if $\pi_B(\alpha)$ is lower than $\pi_B(0)$. To manage this task, we adjust the price $\pi_B(\alpha)$ with respect to $\pi_B^*(\alpha)$ evaluated in section 3.2 in the form $\pi_B(\alpha) = \pi_B^*(\alpha) + \varepsilon$. Using the same technique for the ask price, we get :

$$s(\alpha) = s^*(\alpha) - 2\varepsilon$$

Definition 12 *Let $M^{*}(\alpha, v, w)$ defined as :*

$$\begin{aligned} M^{*}(\alpha, v) &= \alpha\left(\mu_X - \pi_B(0)\right) + v\left(\mu_X - \pi_B^{*}(\alpha)\right) - a\sigma_X^2 (v+\alpha)^2 \\ N^{*}(\alpha, w) &= \alpha\left(\mu_X - \pi_B(0)\right) - w\left(\mu_X - \pi_A^{*}(\alpha)\right) - a\sigma_X^2 (\alpha - w)^2 \end{aligned}$$

$\pi_B^{*}(\alpha)$ and $\pi_A^{*}(\alpha)$ are the solutions of the following equations :

$$\begin{aligned} \frac{1}{\alpha_M}\int_0^{\alpha_M} M^{*}(\alpha, v)dv &= 0 \\ \frac{1}{\alpha_M}\int_0^{\alpha_M} N^{*}(\alpha, v)dv &= 0 \end{aligned}$$

This definition leads to the following prices.

Proposition 13

$$\begin{aligned} \pi_B^{*}(\alpha) &= \mu_X - \frac{2a\sigma_X^2}{\alpha_M}\left[\frac{\alpha_M^2}{3} + \alpha^2 + \frac{\alpha\alpha_M}{2}\left[\frac{2}{3} + \lambda\right]\right] \\ \pi_A^{*}(\alpha) &= \mu_X + \frac{2a\sigma_X^2}{\alpha_M}\left[\frac{\alpha_M^2}{3} + \alpha^2 - \frac{\alpha\alpha_M}{2}\left[\frac{10}{3} - \lambda\right]\right] \\ \varepsilon &= \frac{a\sigma_X^2}{\lambda\alpha_M}\left[\alpha\alpha_M\left(\lambda^2 - \frac{4}{3}\lambda + \frac{1}{6}\right) + \alpha^2(2\lambda - 1) + \alpha_M^2\frac{\lambda^2}{2}\right] \end{aligned}$$

Proof : see the appendix

Even if the formulation is quite involved, the qualitative result is intuitive ; the bid price $\pi_B^{*}(\alpha)$ is lower than $\pi_B^{*}(0)$. Moreover, the difference between the two is given by :

$$\pi_B^{*}(0) - \pi_B^{*}(\alpha) = \frac{2a\sigma_X^2}{\alpha_M}\left[\alpha^2 + \frac{\alpha\alpha_M}{2}\left[\frac{2}{3} + \lambda\right]\right]$$

It is increasing with α, then proving that the bid price is a decreasing function of the inventory.

6 Concluding remarks

In this paper, we have analyzed the relationship between bid and ask prices and inventory in a general framework, in the sense that the market maker faces several sources of uncertainty concerning the direction and the volume of the next trade. It was achieved in the context of linear pricing, that is in a market where unit prices are posted by the market maker. Consequently, the approach is different from the one developed by Eeckhoudt-Roger (1999) who deal with non-linear prices.

However, some of the results have the same qualitative nature because, the last section shows that the relationship between spread and inventory is not always monotonic.

Our model also allows simultaneous orders on the two sides and random volumes; it then generalizes the Ho and Stoll (1983) approach. This point is important because a low-volume trade let the market-maker with a surplus. He then can rise his bid price and decrease his ask price for the following transaction, inducing a complex evolution of spreads with respect to inventory. It is so because we set the ”reference point” to the initial utility of the MM; an alternative approach would be to consider a martingale process for the *ex post* expected utility; however, it is well known that such a process crosses the bounds of any interval in a finite time, almost surely. It means that, in doing so, the MM can experience ”bad trajectories” on which his expected utility deteriorates through successive transactions. The comparison between the two behaviors will be analyzed in a subsequent work.

The other important direction for further research concerns competition between market-makers; our approach, applied to the Ho and Stoll framework, could lead to interesting results related to the determination of equilibria when volumes and the direction of the trade are random. Especially, the cases analyzed by Biais *et al. (1998)* could be considered as special cases of the general uncertainty considered in this paper.

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Appendix

Proof of proposition 8 : The three terms of the integral will be evaluated separately.

$$A_v = \frac{1}{\alpha_M^2} \int_0^{\alpha_M} \int_0^{\alpha_M} (1-\lambda) K(v,0)\, dv dw = \frac{1-\lambda}{\alpha_M} \int_0^{\alpha_M} K(v,0)\, dv$$

$$\begin{aligned} A_v &= \frac{1-\lambda}{\alpha_M} \int_0^{\alpha_M} \left(v(\mu_X - \pi_B) - a v^2 \sigma_X^2 \right) dv && (10) \\ &= \frac{1-\lambda}{\alpha_M} \left[(\mu_X - \pi_B) \frac{\alpha_M^2}{2} - \frac{a \alpha_M^3}{3} \sigma_X^2 \right] \\ &= (1-\lambda) \left[(\mu_X - \pi_B) \frac{\alpha_M}{2} - \frac{a \alpha_M^2}{3} \sigma_X^2 \right] && (11) \end{aligned}$$

$$\begin{aligned} A_w &= \frac{1}{\alpha_M^2} \int_0^{\alpha_M} \int_0^{\alpha_M} (1-\lambda) K(0,w)\, dv dw && (12) \\ &= \frac{1-\lambda}{\alpha_M} \int_0^{\alpha_M} \left(w(\pi_A - \mu_X) - a w^2 \sigma_X^2 \right) dw \\ &= (1-\lambda) \left[(\pi_A - \mu_X) \frac{\alpha_M}{2} - \frac{a \alpha_M^2}{3} \sigma_X^2 \right] \end{aligned}$$

$$\begin{aligned} A_{v,w} &= \frac{1}{\alpha_M^2} \int_0^{\alpha_M} \int_0^{\alpha_M} \lambda K(v,w)\, dv dw \\ &= \frac{\lambda}{\alpha_M^2} \int_0^{\alpha_M} \int_0^{\alpha_M} \left(v(\mu_X - \pi_B) + w(\pi_A - \mu_X) - a(v-w)^2 \sigma_X^2 \right) dv dw \\ &= \frac{\lambda}{\alpha_M^2} \int_0^{\alpha_M} \left(\frac{\alpha_M^2}{2} (\mu_X - \pi_B) + \alpha_M w (\pi_A - \mu_X) - a \frac{(\alpha_M - w)^3 + w^3}{3} \sigma_X^2 \right) dw \\ &= \frac{\lambda \alpha_M}{2} (\pi_A - \pi_B) - \frac{\lambda a \sigma_X^2}{\alpha_M^2} \int_0^{\alpha_M} \left(\frac{(\alpha_M - w)^3 + w^3}{3} \right) dw \\ &= \frac{\lambda \alpha_M}{2} (\pi_A - \pi_B) - \frac{\lambda a \sigma_X^2}{\alpha_M^2} \left[\frac{-(\alpha_M - w)^4 + w^4}{12} \right]_0^{\alpha_M} \\ &= \frac{\lambda \alpha_M}{2} (\pi_A - \pi_B) - \frac{\lambda a \sigma_X^2 \alpha_M^2}{6} = \frac{\lambda \alpha_M}{2} \left((\pi_A - \pi_B) - \frac{a \sigma_X^2 \alpha_M}{3} \right) \end{aligned}$$

We finally obtain the spread as the solution of the following equation :

$$\begin{aligned} A_v + A_w + A_{v,w} &= \frac{\pi_A - \pi_B}{2}\alpha_M - (1-\lambda)\frac{2a\alpha_M^2}{3}\sigma_X^2 - \frac{\lambda a\sigma_X^2\alpha_M^2}{6} \\ &= \frac{\pi_A - \pi_B}{2}\alpha_M - a\alpha_M^2\sigma_X^2\left[(1-\lambda)\frac{2}{3} + \frac{\lambda}{6}\right] \end{aligned}$$

Simplifying by α_M, we get the desired result:

$$s(\lambda) = \pi_A - \pi_B = a\alpha_M\sigma_X^2\left[\frac{4}{3} - \lambda\right]$$

Proof of proposition 11

After a trade of volume α, the new reservation prices are determined by the general formulation of the utility function given in equation 8 in which we add the inventory α traded at a price $\pi_B(0)$ .Using the same type of notations as in the preceding subsection, we define :

$$\begin{aligned} A_v &= \frac{1}{\alpha_M^2}\int_0^{\alpha_M}\int_0^{\alpha_M}\lambda(1-\lambda)M(\alpha, v, 0)dv \\ A_w &= \frac{1}{\alpha_M^2}\int_0^{\alpha_M}\int_0^{\alpha_M}\lambda(1-\lambda)M(\alpha, 0, w)dw \\ A_{vw} &= \frac{1}{\alpha_M^2}\int_0^{\alpha_M}\int_0^{\alpha_M}\lambda^2 M(\alpha, v, w)\,dvdw \\ A_{00} &= (1-\lambda)^2 M(\alpha, 0, 0) \end{aligned}$$

We can now evaluate the four components.

1) Evaluation of A_v

$$A_v = \frac{\lambda(1-\lambda)}{\alpha_M}\int_0^{\alpha_M}\left[\alpha\frac{a\alpha_M\sigma_X^2}{2}\left[\frac{4}{3} - \lambda\right] + v\left(\mu_X - \pi_B(\alpha)\right) - a\sigma_X^2(v+\alpha)^2\right]dv$$

Let $s = a\alpha_M\sigma_X^2\left[\frac{4}{3} - \lambda\right]$; A_v can be rewritten as :

$$\begin{aligned} A_v &= \lambda(1-\lambda)\left[\frac{\alpha s}{2} + \frac{\alpha_M}{2}\left(\mu_X - \pi_B(\alpha)\right) - \frac{a\sigma_X^2}{3\alpha_M}\left((\alpha_M+\alpha)^3 - \alpha^3\right)\right] \\ &= \lambda(1-\lambda)\left[\frac{\alpha s}{2} + \frac{\alpha_M}{2}\left(\mu_X - \pi_B(\alpha)\right) - \frac{a\sigma_X^2}{3}\left(\alpha_M^2 + 3\alpha^2 + 3\alpha_M\alpha\right)\right] \\ &= \lambda(1-\lambda)\left[\frac{\alpha a\alpha_M\sigma_X^2}{2}\left(\frac{4}{3} - \lambda\right) + \frac{\alpha_M}{2}\left(\mu_X - \pi_B(\alpha)\right) - \frac{a\sigma_X^2}{3}\left(\alpha_M^2 + 3\alpha^2 + 3\alpha_M\alpha\right)\right] \\ &= \lambda(1-\lambda)\left[a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(-\frac{2}{3} - \lambda\right) - \frac{1}{3}\left(\alpha_M^2 + 3\alpha^2\right)\right] + \frac{\alpha_M}{2}\left(\mu_X - \pi_B(\alpha)\right)\right] \end{aligned}$$

2) Evaluation of A_w

$$
\begin{aligned}
A_w &= \frac{\lambda(1-\lambda)}{\alpha_M}\int_0^{\alpha_M}\frac{\alpha s}{2} - w\left(\mu_X - \pi_A(\alpha)\right) - a\sigma_X^2 (w-\alpha)^2 dw \\
&= \lambda(1-\lambda)\left[\frac{\alpha s}{2} - \frac{\alpha_M}{2}\left(\mu_X - \pi_A(\alpha)\right) - \frac{a\sigma_X^2}{3}\left(\alpha_M^2 + 3\alpha^2 - 3\alpha\alpha_M\right)\right] \\
&= \lambda(1-\lambda)\left[\frac{\alpha a\alpha_M\sigma_X^2}{2}\left(\frac{4}{3}-\lambda\right) - \frac{\alpha_M}{2}\left(\mu_X - \pi_A(\alpha)\right) - \frac{a\sigma_X^2}{3}\left(\alpha_M^2 + 3\alpha^2 - 3\alpha\alpha_M\right)\right] \\
&= \lambda(1-\lambda)\left[a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right) - \frac{a\sigma_X^2}{3}\left(\alpha_M^2 + 3\alpha^2 - 3\alpha\alpha_M\right)\right] - \frac{\alpha_M}{2}\left(\mu_X - \pi_A(\alpha)\right)\right] \\
&= \lambda(1-\lambda)\left[a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{10}{3}-\lambda\right) - \frac{1}{3}\left(\alpha_M^2 + 3\alpha^2\right)\right] - \frac{\alpha_M}{2}\left(\mu_X - \pi_A(\alpha)\right)\right]
\end{aligned}
$$

3)Evaluation of $A_{v,w}$

$$
\begin{aligned}
A_{vw} &= \frac{1}{\alpha_M^2}\int_0^{\alpha_M}\int_0^{\alpha_M}\lambda^2\left[\frac{\alpha s}{2} + v\left(\mu_X - \pi_B(\alpha)\right) - w\left(\mu_X - \pi_A(\alpha)\right)) - a\sigma_X^2\left(v+\alpha-w\right)^2\right] dv dw \\
&= \frac{\lambda^2}{\alpha_M}\int_0^{\alpha_M}\left[\frac{\alpha s}{2} + \frac{\alpha_M}{2}\left(\mu_X - \pi_B(\alpha)\right) - w\left(\mu_X - \pi_A(\alpha)\right)) - a\sigma_X^2\frac{(\alpha_M+\alpha-w)^3 - (\alpha-w)^3}{3\alpha_M}\right] d \\
&= \lambda^2\left[\frac{\alpha s}{2} + \frac{\alpha_M}{2}\left(\pi_A(\alpha) - \pi_B(\alpha)\right) - a\sigma_X^2\left[\frac{-\alpha^4 + (\alpha_M+\alpha)^4 + (\alpha-\alpha_M)^4 - \alpha^4}{12\alpha_M^2}\right]\right] \\
&= \lambda^2\left[\frac{\alpha s}{2} + \frac{\alpha_M}{2}\left(\pi_A(\alpha) - \pi_B(\alpha)\right) - a\sigma_X^2\left[\frac{\alpha_M^2 + 6\alpha^2}{6}\right]\right] \\
&= \lambda^2\left[a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right) - \left[\frac{\alpha_M^2 + 6\alpha^2}{6}\right]\right] + \frac{\alpha_M}{2}\left(\pi_A(\alpha) - \pi_B(\alpha)\right)\right]
\end{aligned}
$$

4) Evaluation of A_{00}

$$
\begin{aligned}
A_{00} &= (1-\lambda)^2\left[E_X\left[U(W_0 + \alpha\left(X - \pi_B(0)\right)\right] - U(W_0)\right] \\
&= (1-\lambda)^2\left(\frac{\alpha s}{2} - a\sigma_X^2\alpha^2\right) \\
&= (1-\lambda)^2\left(\frac{\alpha a\alpha_M\sigma_X^2}{2}\left(\frac{4}{3}-\lambda\right) - a\sigma_X^2\alpha^2\right) \\
&= (1-\lambda)^2 a\sigma_X^2\alpha\left[\frac{1}{2}\alpha_M\left(\frac{4}{3}-\lambda\right) - \alpha\right]
\end{aligned}
$$

We can now evaluate $\Phi = A_v + A_w + A_{vw} + A_{00}$:

$$
\Phi = \lambda(1-\lambda)\left[a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(-\frac{2}{3}-\lambda\right) - \frac{1}{3}\left(\alpha_M^2 + 3\alpha^2\right)\right] + \frac{\alpha_M}{2}\left(\mu_X - \pi_B(\alpha)\right)\right]
$$

$$+\lambda(1-\lambda)\left[a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{10}{3}-\lambda\right)-\frac{1}{3}\left(\alpha_M^2+3\alpha^2\right)\right]-\frac{\alpha_M}{2}\left(\mu_X-\pi_A(\alpha)\right)\right]$$
$$+(1-\lambda)^2 a\sigma_X^2\alpha\left[\frac{1}{2}\alpha_M\left(\frac{4}{3}-\lambda\right)-\alpha\right]$$
$$+\lambda^2\left[a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right)-\left[\frac{\alpha_M^2+6\alpha^2}{6}\right]\right]+\frac{\alpha_M}{2}\left(\pi_A(\alpha)-\pi_B(\alpha)\right)\right]$$

$$\begin{aligned}\Phi &= \frac{\lambda\alpha_M}{2}\left(\pi_A(\alpha)-\pi_B(\alpha)\right)\\ &+\lambda(1-\lambda)a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(-\frac{2}{3}-\lambda\right)-\frac{1}{3}\left(\alpha_M^2+3\alpha^2\right)\right]\\ &+\lambda(1-\lambda)a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{10}{3}-\lambda\right)-\frac{1}{3}\left(\alpha_M^2+3\alpha^2\right)\right]\\ &+(1-\lambda)^2 a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right)-\alpha^2\right]\\ &+\lambda^2 a\sigma_X^2\left[\frac{\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right)-\left[\frac{\alpha_M^2+6\alpha^2}{6}\right]\right]\end{aligned}$$

$$\begin{aligned}\Phi &= \frac{\lambda\alpha_M}{2}\left(\pi_A(\alpha)-\pi_B(\alpha)\right)+\frac{a\sigma_X^2\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right)\\ &-\lambda(1-\lambda)a\sigma_X^2\left[\frac{2}{3}\left(\alpha_M^2+3\alpha^2\right)\right]\\ &-(1-\lambda)^2 a\sigma_X^2\alpha^2-\lambda^2 a\sigma_X^2\left[\frac{\alpha_M^2+6\alpha^2}{6}\right]\end{aligned}$$

$$\begin{aligned}\Phi &= \frac{\lambda\alpha_M}{2}\left(\pi_A(\alpha)-\pi_B(\alpha)\right)+\frac{a\sigma_X^2\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right)\\ &-a\sigma_X^2\left[\lambda(1-\lambda)\left[\frac{2}{3}\left(\alpha_M^2+3\alpha^2\right)\right]+(1-\lambda)^2\alpha^2+\lambda^2\left[\frac{\alpha_M^2+6\alpha^2}{6}\right]\right]\end{aligned}$$

We then deduce the spread :

$$\pi_A(\alpha)-\pi_B(\alpha)=\frac{2a\sigma_X^2}{\lambda\alpha_M}\left[\lambda\alpha_M^2\left[\frac{2}{3}-\frac{\lambda}{2}\right]+\alpha^2-\frac{\alpha\alpha_M}{2}\left(\frac{4}{3}-\lambda\right)\right]$$

Proof of proposition 13

The bid price is obtained following the usual way :

$$\frac{1}{\alpha_M}\int_0^{\alpha_M} M^*(\alpha,v)dv = \frac{1}{\alpha_M}\int_0^{\alpha_M}\left(\frac{\alpha a\alpha_M\sigma_X^2}{2}\left[\frac{4}{3}-\lambda\right]+v\left(\mu_X-\pi_B^*(\alpha)\right)-a\sigma_X^2\left(v+\alpha\right)^2\right)dv$$

$$= \frac{\alpha a\alpha_M\sigma_X^2}{2}\left[\frac{4}{3}-\lambda\right]+\frac{\alpha_M}{2}\left(\mu_X-\pi_B^*(\alpha)\right)-a\sigma_X^2\frac{(\alpha_M+\alpha)^3-\alpha^3}{3\alpha_M}$$

We obtain directly the bid price by the following transformations :

$$\begin{aligned}
\mu_X-\pi_B^*(\alpha) &= \frac{2a\sigma_X^2}{\alpha_M}\left[\frac{(\alpha_M+\alpha)^3-\alpha^3}{3\alpha_M}-\frac{\alpha\alpha_M}{2}\left[\frac{4}{3}-\lambda\right]\right] \\
&= \frac{2a\sigma_X^2}{\alpha_M}\left[\frac{\left(\alpha_M^2+3\alpha\alpha_M+3\alpha^2\right)}{3}-\frac{\alpha\alpha_M}{2}\left[\frac{4}{3}-\lambda\right]\right] \\
&= \frac{2a\sigma_X^2}{\alpha_M}\left[\frac{\alpha_M^2}{3}+\alpha\alpha_M+\alpha^2-\frac{\alpha\alpha_M}{2}\left[\frac{4}{3}-\lambda\right]\right] \\
&= \frac{2a\sigma_X^2}{\alpha_M}\left[\frac{\alpha_M^2}{3}+\alpha^2+\frac{\alpha\alpha_M}{2}\left[\frac{2}{3}+\lambda\right]\right]
\end{aligned}$$

The same calculations concerning the ask price lead to :

$$\frac{1}{\alpha_M}\int_0^{\alpha_M} N^*(\alpha,v)dv = \frac{\alpha a\alpha_M\sigma_X^2}{2}\left[\frac{4}{3}-\lambda\right]-\frac{\alpha_M}{2}\left(\mu_X-\pi_A^*(\alpha)\right)-a\sigma_X^2\frac{(\alpha_M-\alpha)^3+\alpha^3}{3\alpha_M}$$

From which we deduce the ask price :

$$\begin{aligned}
\frac{\alpha_M}{2}\left(\mu_X-\pi_A^*(\alpha)\right) &= \frac{\alpha a\alpha_M\sigma_X^2}{2}\left[\frac{4}{3}-\lambda\right]-a\sigma_X^2\frac{(\alpha_M-\alpha)^3+\alpha^3}{3\alpha_M} \\
&= -a\sigma_X^2\left(-\alpha\alpha_M\left[\frac{5}{3}-\frac{\lambda}{2}\right]+\frac{\alpha_M^2}{3}+\alpha^2\right)
\end{aligned}$$

$$\pi_A^*(\alpha) = \mu_X+\frac{2a\sigma_X^2}{\alpha_M}\left[-\alpha\alpha_M\left[\frac{5}{3}-\frac{\lambda}{2}\right]+\frac{\alpha_M^2}{3}+\alpha^2\right]$$

The spread is then :

$$\pi_A^*(\alpha)-\pi_B^*(\alpha) = \frac{2a\sigma_X^2}{\alpha_M}\left[\alpha\alpha_M\left(\lambda-\frac{4}{3}\right)+\frac{2\alpha_M^2}{3}+2\alpha^2\right]$$

We can now calculate ε.

$$s^*(\alpha)-s(\alpha) = \frac{2a\sigma_X^2}{\alpha_M}\left[\alpha\alpha_M\left(\lambda-\frac{4}{3}\right)+\frac{2\alpha_M^2}{3}+2\alpha^2\right]-\frac{2a\sigma_X^2}{\lambda\alpha_M}\left[\alpha^2+\lambda\alpha_M^2\left(\frac{2}{3}-\frac{\lambda}{2}\right)-\alpha\alpha_M\left(\frac{2}{3}-\frac{\lambda}{2}\right.\right.$$

$$
\begin{aligned}
&= \frac{2a\sigma_X^2}{\alpha_M}\left[\alpha\alpha_M\left(\lambda-\frac{4}{3}\right)+\frac{2\alpha_M^2}{3}+2\alpha^2-\frac{\alpha^2}{\lambda}-\alpha_M^2\left(\frac{2}{3}-\frac{\lambda}{2}\right)+\alpha\alpha_M\left(\frac{2}{3\lambda}-\frac{1}{2\lambda}\right)\right] \\
&= \frac{2a\sigma_X^2}{\alpha_M}\left[\alpha\alpha_M\left(\lambda-\frac{4}{3}+\frac{1}{6\lambda}\right)+\alpha^2\left(2-\frac{1}{\lambda}\right)+\alpha_M^2\frac{\lambda}{2}\right]
\end{aligned}
$$

This leads to :

$$
\varepsilon=\frac{a\sigma_X^2}{\lambda\alpha_M}\left[\alpha\alpha_M\left(\lambda^2-\frac{4}{3}\lambda+\frac{1}{6}\right)+\alpha^2(2\lambda-1)+\alpha_M^2\frac{\lambda^2}{2}\right]
$$

The prices π_A and π_B are now given by :

$$
\begin{aligned}
\pi_B(\alpha) &= \mu_X-\frac{2a\sigma_X^2}{\alpha_M}\left[\frac{\alpha_M^2}{3}+\alpha^2+\frac{\alpha\alpha_M}{2}\left[\frac{2}{3}+\lambda\right]\right]+ \\
&\quad \frac{a\sigma_X^2}{\alpha_M}\left[\alpha\alpha_M\left(\lambda-\frac{4}{3}+\frac{1}{6\lambda}\right)+\alpha^2\left(2-\frac{1}{\lambda}\right)+\alpha_M^2\frac{\lambda}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \mu_X-\frac{2a\sigma_X^2}{\alpha_M}\left[\frac{\alpha_M^2}{3}+\alpha^2+\frac{\alpha\alpha_M}{2}\left[\frac{2}{3}+\lambda\right]\right]+ \\
&\quad \frac{a\sigma_X^2}{\alpha_M}\left[\alpha\alpha_M\left(\lambda-\frac{4}{3}+\frac{1}{6\lambda}\right)+\alpha^2\left(2-\frac{1}{\lambda}\right)+\alpha_M^2\frac{\lambda}{2}\right] \\
&= \mu_X-\frac{a\sigma_X^2}{\alpha_M}\left[\frac{2\alpha_M^2}{3}+2\alpha^2+\alpha\alpha_M\left[\frac{2}{3}+\lambda\right]-\alpha\alpha_M\left(\lambda-\frac{4}{3}+\frac{1}{6\lambda}\right)-\alpha^2\left(2-\frac{1}{\lambda}\right)-\alpha_M^2\frac{\lambda}{2}\right] \\
&= \mu_X-\frac{a\sigma_X^2}{\alpha_M}\left[\frac{\alpha^2}{2}-\alpha\alpha_M\left(\frac{1}{6\lambda}-2\right)-\alpha_M^2\left(\frac{\lambda}{2}-\frac{2}{3}\right)\right]
\end{aligned}
$$

With similar calculations, we obtain the ask price.

$$
\begin{aligned}
\pi_A(\alpha) &= \mu_X+\frac{2a\sigma_X^2}{\alpha_M}\left[-\alpha\alpha_M\left[\frac{5}{3}-\frac{\lambda}{2}\right]+\frac{\alpha_M^2}{3}+\alpha^2\right] \\
&\quad -\frac{a\sigma_X^2}{\alpha_M}\left[\alpha\alpha_M\left(\lambda-\frac{4}{3}+\frac{1}{6\lambda}\right)+\alpha^2\left(2-\frac{1}{\lambda}\right)+\alpha_M^2\frac{\lambda}{2}\right] \\
\pi_A(\alpha) &= \mu_X+\frac{a\sigma_X^2}{\alpha_M}\left[\frac{\alpha^2}{\lambda}-\alpha\alpha_M\left(2+\frac{1}{6\lambda}\right)-\alpha_M^2\left(\frac{\lambda}{2}-\frac{2}{3}\right)\right]
\end{aligned}
$$