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Beneficial Risk-sharing without Assignment of Loss Probabilities

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1. Introduction

The "diversification theorem" says that if a risk-averse individual has a choice between two assets with identical but random returns, the individual prefers to invest half of the endowment in each asset (Rothschild and Stiglitz, 1971, p83). This theorem is basic in portfolio theory. Nonetheless, its institutional and decision-theoretical implications have not been fully observed. The theorem may be restated in the following way: Two identical risk-averse individuals with assets of the same value and distribution of returns gain by sharing the return.[1] Importance here is that the restated theorem is true for *any* distribution of returns, as long as the distribution is the *same* for both parties, e.g., the probability of a certain loss may be 0.1 or 0.0001 and the parties gain by sharing in both cases.

The revised theorem implies that risk-sharing is mutually beneficial also when the probability is unknown, unpredictable or genuinely uncertain, given that the risk is the same for both. In fact, it is not necessary that the risk is the same for both, it suffices that the parties accept a *presumption of equality*. In other words, the parties may share losses beneficially after they have accepted that there is no reason to assume that their probabilities of losses differ. This implies that mutual sharing of highly uncertain losses may be superior to insurance that requires a premium fixed *ex ante*.

An interesting theoretical implication of the revised theorem is that equals facing uncertainty may share losses without addressing probabilities to potential outcomes. That is, instead of assessing probabilities to uncertain outcomes that, in fact, cannot be estimated, "equal risks" may be identified through measurable units. For instance, a marine mutual pool of ship-owners with equal value of ship, and the same probability of loss (requiring the same

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[1] Theorem 1 in Section 3 is served as a restatement.

quality of crew and ship, and similar route) may share losses beneficially also in the absence of actuarial information.

In "Risk-Sharing Institutions for Unpredictable Losses" G. Skogh (1999)[2] shows the findings mentioned above. The article is, however, limited to the case with two identical pool members. An obvious question is: Does sharing work similarly when the risks and individuals differ? In the present article we examine the conditions for beneficial risk-sharing in a world where individuals and risks differ. First, in Section 2, we present a ship-owner example to illustrate the practical relevance of the following analysis. The diversification theorem is restated in Section 3. We prove that equal sharing is a Nash solution and it is thus Pareto optimal, if the two individuals have the same utility function and face the same distribution of losses. Section 4 shows that a Nash solution requires that the sharing vary according to the risk-aversion of the parties. Equal sharing is a mutual improvement, as compared with no sharing at all. Section 5 analyses mutual sharing when the individuals´ losses differ in probabilities, or in amount. We obtain a risk-sharing solution, although it cannot be proved as a Nash solution, that makes individuals better off and, importantly, requires no information on actual values of probabilities. Section 6 ends the article with concluding remarks.

2. The Ship Owner Tale

Once upon a time there was a risk-averse ship-owner looking for someone willing to share potential loss of cargo and ship. No insurance was available. That is, there was no one willing to accept the risk for a premium *ex ante.* On the other hand, there was another risk-averse ship owner with a similar ship, cargo, crew and route. The two ship owners had no idea about the probability of the ships not returning back. Yet, due to their similarities, they could agree that it was reasonable to assume that the probability of a wrecked ship was the same for the two ships. Under the presumptions of the same probability of a loss and of the same amount of a loss, they realised that they could expect to benefit by sharing a loss of ship. And so they did.

The two ship owners also realised that the pooling would be more efficient if they were more partners in the risk-sharing group. There were several other ship owners, but they were different. Some ships were of poor quality, and some were loaded with expensive cargo. Most journeys were short but some were long. Moreover, the skill of the crew varied. Thus the potential losses and the probabilities of a loss of ship varied. Consequently, a prior assumption on equal risk was unrealistic. So was an equal share of a loss. Thus, a sharing of losses was not at once feasible.

Nevertheless, our risk-averse ship owners continued to look for extended risk sharing. They realised that a generally accepted presumption of equal risk required equality in *identifiable* indicators such as size of ship, cargo, route, etc. A first measure to make sharing possible was, therefore, to *equalise* ships, crews, route etc. The (unobservable) probability of losses was hereby unified. The offer to join the pool was, therefore, restricted to ship owners with the same cargo and destination that could show similar quality of ship and crew. Renovation of ships and education of the crew was undertaken, and the pool was hereby extended to several ships, but still with the same cargo and destination that could show similar quality of ship and crew.

A limitation in the pooling was the varying value of ship and cargo and varying destination. Some ships were big and some small. Some cargo was of small value but sometimes the value was large. Some travelled short and some long. This shortcoming was solved in the following way: They first defined a unit of measure, named a "share", and then they allowed people to join the pool with different shares. For example, the smallest value of the cargo in the smallest ship travelling a nautical mile was defined as a share. And thus a ship of double value (cargo included) travelling a nautical mile could be treated as two shares and so on. Consequently, a ten-share ship was treated as ten one share ships. That is, the ten-share ship had to contribute with ten shares at a loss.

A large number of ship owners joined the pool since they regarded the membership as beneficial and reasonably fair. In this way, the risks at sea were diversified. The member of the pool earned a relatively stable income and became wealthy, especially after some further refinements. First, the pool introduced prepayments proportional to the shares. The prepayments was funded and used for coverage of losses. Hereby a diversification over time was established. Since the pool members had the common interest in prevention of accidents, they introduced safety regulations according to the information available. As time went on they also obtained further information on "high" and "low" risks. The tendency of low risks to leave the pool was mitigated by adjustments in the shares. The benefit of a large pool was hereby maintained. Later, when information of the risks was well established insurers offering coverage of losses at a premium fixed *ex ante* appeared in the market.

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[2] See also Skogh (1998) that discusses insurability of new (uninsurable) industrial hazards.

3. The Basic Model and the Diversification Theorem Restated

The tale above is, we claim, a plausible story of how risk averse traders may handle a situation with genuine uncertainty or statistically unpredictable losses, without addressing arbitrary probabilities to the possible outcomes. In the rest on the paper we present a mathematical model that clarifies the problem and its possible solutions.

Suppose that there are two individuals, who have increasing and strictly concave utility functions: $u_i(\cdot)$ (i = 1, 2): $u_i'(\cdot) > 0$ and $u_i''(\cdot) < 0$. Individuals 1 and 2 are assumed to have the same endowment w. The assumption of the same endowment avoids an individual's income effect on a sharing solution. To avoid an individual's insolvency problem, we further assume that w is large enough so that any of the individuals can afford to share any amount of a total loss of the two individuals.[3]

Assume that individual i (i = 1, 2) faces a possible loss, denoted by a random variable L_i. L_i's (i = 1, 2) satisfy the Bernoulli distribution and is distributed independently, but not necessary identically: the random variable L_i (i = 1, 2) is equal to l_i with a probability p_i and equal to 0 with a probability $1-p_i$.

Although we assume a Bernoulli distribution for L_i's, the exact value of probability p_i may be unknown. So this assumption is not against the story that information on probabilities may be unavailable.

With the assumptions, individual i (i = 1, 2) has an initial portfolio $x_i = w - L_i$. And the total loss of the two individuals will be $\sum_{i=1}^{2} L_i = L_1 + L_2$. The two individuals come together, and look for a sharing solution that make both of them better off. Suppose that the two individuals share the total loss according to $S = (s, 1-s)'$, $0 \le s \le 1$, where s denotes the share of individual 1. $1-s$ is obviously the proportion of individual 2 sharing the total loss.[4] Thus, the two individuals will end with a final portfolio $y_1 = w - s\sum_{i=1}^{2} L_i$ and $y_2 = w - (1-s)\sum_{i=1}^{2} L_i$ respectively.

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[3] Wu (2001 and 2002) models a mutual corporative game in a more general basis.

[4] This type of sharing rule is called *equal-proportion-share* in Wu (2001). According to the definition of the equal-proportion-share, each individual in the pool shares all others' losses in an equal proportion: Individuals agree with a share rule $S = (s_1, s_2, \ldots, s_n)'$ in advance, where s_i denotes a proportion of individual i (i = 1, 2, …, n) sharing the total loss, and then put all their losses into the pool. When losses occur, they share the total loss of the pool, based on S, and do not care about who experiences the losses.

Before we continue the model, we note that the final portfolio y_i is totally decided by a share solution $(s, 1-s)'$. This assumption is consistent with the tale where there is no insurer available, no fixed payment, and no side-payment. This differs from Borch's and Aase's (2002) models in which they assume a final portfolio Y_i and allow for a non-zero side-payment d_i in Y_i. According to their model, it could be that $y_1 = w - d_1 - s\sum_{i=1}^{2} L_i$ and $y_2 = w - d_2 - (1-s)\sum_{i=1}^{2} L_i$. We obviously have a different definition of y_i.

The two individuals will bargain with each other to find a sharing solution. Individual i enters the game with an initial portfolio x_i, exchanges it with the other individual, and ends with a final portfolio y_i. Obviously, individuals will enter the game only if they can obtain no less expected utilities than by not joining it, i.e., $EU_i(y_i) \geq EU_i(x_i)$ for i = 1, 2, where $EU_i(\cdot)$ denotes individual i's expected utility function. $EU_i(x_i)$, denoted by r_i in below, is individual i's reservation utility. A portfolio y_i satisfying $EU_i(y_i) \geq EU_i(x_i)$ for all i (i = 1, 2) can thus be called a feasible portfolio (solution). It is a solution making no individual worse off. This bargaining game can be denoted by a combination (B, d), in which

$$\mathrm{B} = \left\{ (EU_i)_{i=1,2} \middle| EU_i = EU_i(w - s_i \sum_{i=1,2} L_i); s_1 = s, s_2 = 1-s, 0 \leq s \leq 1 \right\}$$

and

$$d = \left\{ (EU_i)_{i=1,2} \middle| EU_i = EU_i(w - L_i) \right\}.$$

Solving this bargaining game entails finding a proper s such that $(EU_i)_{i=1,2} \in \mathrm{B}$ are accepted by both of the two individuals.

More specifically, if both of them do not join any pool, then their expected utility functions (reservation utilities) will respectively be

$$r_1 = p_1 u_1(w-l_1) + (1-p_1)u_1(w)$$

and

$$r_2 = p_2 u_2(w-l_2) + (1-p_2)u_2(w)$$

If the two individuals share the total loss according to $S = (s, 1-s)'$, $0 \leq s \leq 1$, where s is defined before, then they will end with utilities respectively

$$EU_1 = p_1 p_2 u_1(w-s(l_1+l_2)) + p_1(1-p_2)u_1(w-sl_1) + (1-p_1)p_2 u_1(w-sl_2) + (1-p_1)(1-p_2)u_1(w)$$

and

$$EU_2 = p_1p_2u_2(w-(1-s)(l_1+l_2)) + p_1(1-p_2)u_2(w-(1-s)l_1) + (1-p_1)p_2u_2(w-(1-s)l_2) + (1-p_1)(1-p_2)u_2(w)$$

According to the definition of the Nash solution this bargaining game solves for s

$$\begin{aligned} &\underset{0\le s\le 1}{Max}(EU_1 - r_1)(EU_2 - r_2) \\ s.t., \quad &EU_1 \ge r_1 \qquad\qquad \text{(Na)} \\ &EU_2 \ge r_2 \end{aligned}$$

Mathematical derivatives give us the following: $\frac{\partial EU_1}{\partial s} < 0$, $\frac{\partial EU_2}{\partial s} > 0$, $\frac{\partial^2 EU_1}{\partial s^2} < 0$, $\frac{\partial^2 EU_2}{\partial s^2} < 0$ and $\frac{\partial^2 (EU_1 - r_1)(EU_2 - r_2)}{\partial s^2} < 0$. Thus, the constraint set $\{s \in [0, 1] \mid EU_1 \ge r_1, EU_2 \ge r_2\}$ is a closed convex set and, therefore, maximizing program (Na) will have a unique solution, as long as the constraint set is nonempty. Solving the program (Na) requires the use of the Kuhn-Tucker theorem. The Lagrangian will be

$$L = (EU_1 - r_1)(EU_2 - r_2) + \mu_1(EU_1 - r_1) + \mu_2(EU_2 - r_2)$$

in which μ_i (i = 1, 2) is the Lagrange multipliers related to constraint $EU_i \ge r_i$ (i = 1, 2): $\mu_i \ge 0$ and $\mu_i(EU_i - r_i) = 0$ (i = 1, 2). The first order condition requires the derivative of the Lagrangian with respect to s equal to zero, which produces equation

$$(EU_1 - r_1)\frac{\partial EU_2}{\partial s} + (EU_2 - r_2)\frac{\partial EU_1}{\partial s} + \mu_1\frac{\partial EU_1}{\partial s} + \mu_2\frac{\partial EU_2}{\partial s} = 0 \quad (1)$$

When both $EU_1 > r_1$ and $EU_2 > r_2$, equation (1) becomes

$$(EU_1 - r_1)\frac{\partial EU_2}{\partial s} + (EU_2 - r_2)\frac{\partial EU_1}{\partial s} = 0 \qquad (2)$$

The diversification theorem restated

We start with the case where both individuals have the same utility function and the same distribution of losses. The Nash solution is risk sensitive. The assumption of the same utility function eliminates the effects of individuals' differences on the Nash solution. Section 4 will specifically investigate the effects of individuals' utilities on the Nash solution. The assumption of the same distribution of losses means that both the amount of losses and the probability of losses for the two individuals are equal: Random variable L_i is equal to L with probability p_i and equal to 0 with probability $1-p_i$. And $p_1 = p_2 = p$.

Theorem 1 According to the Nash solution, when two individuals have the same utility function and face the same distribution of losses, they share the total loss equally. Following the notations above, the Nash solution of (Na) is that $s = ½$.

Proof: Let $u_i(\cdot) =: u(\cdot)$ for $i = 1, 2$. Obviously, $r_1 = r_2 =: r$. When $s = ½$, $EU_1 = EU_2 =: EU$. Then,

$$\begin{aligned}
&EU - r \\
&= p^2u(w-L) + 2p(1-p)u(w-L/2) + (1-p)^2u(w) - pu(w-L) - (1-p)u(w) \\
&= 2p(1-p)[u(w-L/2) - (u(w-L) + u(w))/2] \\
&> 0
\end{aligned}$$

since $u(\cdot)$ is strictly concave. Hence under the assumptions, the constraint set of (Na) is nonempty and therefore there exists a unique solution. $EU_1 = EU_2 > r$ means that $\mu_1 = \mu_2 = 0$ in Equation (1). Furthermore, $\frac{\partial EU_1}{\partial s} = -\frac{\partial EU_2}{\partial s}$ at $s = ½$ implies that $s = ½$ satisfies equation (2) and therefore becomes the unique Nash solution. #

The theorem can be simply extended to a case where more than two individuals bargaining with each other; If several individuals have the same utility function and face the same distribution of losses, they share the total loss equally.

The diversification theorem restated has an important feature – it is true independent of the probability p, $0< p<1$, assuming p is equal to the pool members. That is, Pareto optimal risk sharing at uncertainty does not necessary require that the parties address a subjective probability of a loss, which is a standard requirement in the decision theory following Savage(1954).

4. When Risk Aversion Varies among Pool Members

In this section, we investigate how a Nash solution is affected by individuals' risk attitudes.

Before we present the model, it is worth making a note that "for any model of bargaining that depends in a non-trivial way on the expected utility function of the bargainers, the underlying assumption is that the risk aversion of the bargainers influences the outcome of bargaining. That is, the risk aversion of the bargainers influences the decisions they make in the course of negotiations, which in turn influence the outcome of bargaining" (Kihlstrom and Roth, 1982). Thus, although this section discusses the effect of the individuals' risk attitudes (individuals' risk aversions) on the bargaining outcome, it is not assumed that the bargainers know one another's risk postures.

Suppose that there are two individuals. Their losses are distributed independently and identically: Both of them face the same amount of possible loss L with the same probability p. However, they have different utility functions: Individual A has an increasing and strictly concave utility function $u(\cdot)$ and individual B has an increasing and strictly concave utility function $v(\cdot)$. Assume that individual A is more risk-averse than individual B, which, according to Pratt (1964), means that, for any x, $R_A(x) > R_B(x)$, where $R_A(x) = -\dfrac{u''(x)}{u'(x)}$ and $R_B(x) = -\dfrac{v''(x)}{v'(x)}$ are individuals A's and B's measures of absolute risk aversions respectively. Or equivalently, individual A is more risk-averse than individual B if and only if there is an increasing and strictly concave function $G(\cdot)$, such that $u(x) = G(v(x))$.

Individuals A's and B's reservation utilities are $r_A = pu(w-L) + (1-p)u(w)$ and $r_B = pv(w-L) + (1-p)v(w)$. By joining a pool in which individual A bears a share of total loss s ($s \in [0, 1]$) and individual B bears the others, i.e. $1-s$, their expected utilities are

$$EU_A = p^2u(w-2sL) + (1-p)^2u(w) + 2p(1-p)u(w-sL)$$

and

$$EU_B = p^2v(w-2(1-s)L) + (1-p)^2v(w) + 2p(1-p)v(w-(1-s)L)$$

Substitute EU_A, EU_B, r_A and r_B for EU_1, EU_2, r_1 and r_2 in (Na) respectively, i.e., individual A corresponds to individual 1 and individual B to individual 2 in Section 3. The first order condition will be

$$(EU_A - r_A)\frac{\partial EU_B}{\partial s} + (EU_B - r_B)\frac{\partial EU_A}{\partial s} + \mu_A\frac{\partial EU_A}{\partial s} + \mu_B\frac{\partial EU_B}{\partial s} = 0 \qquad (1')$$

in which $\mu_i \geq 0$ (i = A, B). And if $EU_i > r_i$, then relative $\mu_i = 0$, for i = A, B.

Obviously, if there is a Nash solution, then the Nash solution has to be in the open interval (0, 1), because $EU_A < r_A$ at $s = 1$, and $EU_B < r_B$ at $s = 0$. Besides, since both $EU_A > r_A$ and $EU_B > r_B$ exist at $s = ½$ from the proof of Theorem 1, $s = ½$ is a feasible solution and the constraint set of (Na) is nonempty. It means that there exists a unique Nash solution and, at the Nash solution s, $EU_A > r_A$ and $EU_B > r_B$. From the Kuhn-Tucker theorem, the Nash solution s should solve

$$(EU_A - r_A)\frac{\partial EU_B}{\partial s} + (EU_B - r_B)\frac{\partial EU_A}{\partial s} = 0 \qquad (2')$$

Take into account an extreme case, in which individual B is risk-neutral and individual A is risk-averse.[5] If this is the case, then $EU_A > r_A$ and $EU_B = r_B$, at $s = ½$. Since $EU_A > r_A$ at $s = ½$, and $EU_A < r_A$ at $s = 1$, there exists $s_0 > ½$ such that $EU_A = r_A$ at s_0. Thus, it must be $s \in [0, s_0]$ to satisfy $EU_A \geq r_A$. In addition, because $EU_B < r_B$ at $s = 0$ and $EU_B = r_B$ at $s = ½$, it must be $s \in [½, 1]$ to satisfy $EU_B \geq r_B$. $s \in [0, s_0]$ and $s \in [½, 1]$ gives the Nash solution $s \in [½, s_0]$. Moreover, when $s \in (½, s_0)$, both $EU_A > r_A$ and $EU_B > r_B$. Therefore, the Nash solution will satisfy $s > ½$.

This extreme case suggests that if both individuals are risk-averse, the more risk-averse individual might bear more of the total loss than the less risk-averse individual. Under the assumptions of the model specified in this section, the Nash solution might satisfy that $s > ½$. Unfortunately, this conjecture can only be confirmed in a special case where individuals' utility functions are quadratic.[6] In the general expected utility approach, the Nash solution will not be necessarily larger than ½ and actually can be any value between 0 and 1.

Theorem 2 Two risk-averse individuals with different degrees of risk aversion face the same distribution of loss. Although the equal-share ($s = ½$) is a feasible solution, the Nash solution may be larger than, equal to, or less than ½ if the two risk-averse individuals maximize the general expected utility functions. In other words, according to the Nash solution, the more risk-averse individual may bear more total loss than, or less than, or the same as, the less risk-averse individual.

Proof: Let $h(s)$ denote the left side of the equation (2′). As $h'(s) < 0$ in the feasible interval, $h(s)$ is a decreasing function of s. If $h(½) > 0$, then the Nash solution solving $h(s) = 0$ will be $s > ½$. And if $h(½) < 0$, then the Nash solution will be $s < ½$. To prove that the sign $h(½)$ is not certain, an example where both $h(½) > 0$ and $h(½) < 0$ appear must be given.

From the expression EU_A and EU_B,

$$\frac{\partial EU_A}{\partial s} = -2pL(pu'(w-2sL)+(1-p)u'(w-sL))$$

$$\frac{\partial EU_B}{\partial s} = 2pL(pv'(w-2(1-s)L)+(1-p)v'(w-(1-s)L))$$

[5] When a risk-neutral individual B is assumed, the Nash solution does not necessarily solve (2′), since, with the assumption, $EU_B = r_B$ at the Nash solution may be the case.

[6] A proof can be found in Wu (2002).

Thus,

$$h(½) = 4p^2(1-p)L\{\left(u(w-L/2)-\frac{u(w-L)+u(w)}{2}\right)\left(pv'(w-L)+(1-p)v'(w-L/2)\right) -$$

$$\left(v(w-L/2)-\frac{v(w-L)+v(w)}{2}\right)\left(pu'(w-L)+(1-p)u'(w-L/2)\right)\}$$

Assume that $v(x) = -1/x$, $G(x) = -e^{-x}$, and $u(x) = G(v(x)) = -e^{1/x}$. One can check that both $v(x)$ and $u(x)$ are increasing and strictly concave functions and $u(x)$ are more concave than $v(x)$. a) If $L = 0$, or if $L << w$ such that $\frac{1}{w-L} \approx \frac{1}{w-L/2} \approx \frac{1}{w}$, then $h(½) = 0$, or $h(½) \approx 0$; b) If $w = 5$, $L = 4$ and $p = ½$, then $h(½) < 0$; c) If $w = 2$, $L = 1$ and $p = ½$, then $h(½) > 0$. The theorem is thus proved. #

The conclusion in Theorem 2 looks indifferent from usual conclusions in risk aversion. Pratt (1964) claimed that, when two individuals have different degrees of risk aversion, a more risk-averse individual would be willing to pay more against a risk than a less risk-averse individual. In addition, the well-known Borch's theory in risk sharing (See, e.g., Aase, 2002) said that in the Pareto optimal set in considering a reciprocal reinsurance treaty, a more risk-averse individual would share more of a total loss in a pool than a less risk-averse individual. A Nash solution in the Borch's model should, of course, be that a more risk-averse individual shares more of a total loss in a pool than a less risk-averse individual.

Our explanations are first that in this mutual bargaining game, the bargaining outcome is a risky outcome. In other words, if two individuals sign a share contract according to Nash, their final portfolios y_i are still random variables related to a total loss. This happens to be consistent with Roth and Rothblum (1982); Although the Nash solution generally predicts that risk aversion is a disadvantage in bargaining, risk aversion does not always have to be a disadvantage, if a bargaining game concerns risky outcomes, as well as riskless outcomes. Secondly, as mentioned, in the Borch's model a non-zero side-payment d_i is allowed. The conclusion that a more risk-averse individual shares more of a total loss than a less risk-averse individual is subject to that the more risk-averse individual is willing to pay a higher side-payment to the less risk-averse individual. Obviously, a more risk-averse individual prefers less risk than a less risk-averse individual. Therefore, he is willing to pay a higher side-payment to get rid of risks. Here, in our case, there is no side-payment. Due to the situation, the more risk-averse individual may share more of a total loss, possibly because he is afraid of losing the pool membership, which could happen if the less risk-aversion individual does not accept to join the pool.

In a real process, do the insureds, who have different levels of risk aversion but join the same mutual cooperative, contribute differently to the same risk? The answer is presumably no. Bargaining costs may exclude an efficient solution. On the other hand, since both $EU_A > r_A$ and $EU_B > r_B$ exist at $s = ½$ from the proof of Theorem 1, $s = ½$ is a feasible solution and makes both individuals strictly better off. That is, $s = ½$ can be a bargaining solution that serves as a mutually beneficial risk-sharing. Thus, equal sharing may have a rational application also among individuals that differ in risk aversion.

5. When Risk Varies among Pool Members

As the story in Section 2 notes, the risk may vary among individuals, which could be an obstacle to sharing. However, pricing of observable differences may solve the problem. To clarify, we assume that two individuals have the same utility function but different distributions of losses. We present a case where the amount of possible loss L is the same, but a high-risk individual faces a higher probability of loss than a low-risk individual does. p_h and p_l denote their probabilities of losses respectively and $p_h > p_l$.[7]

To be close to the tale above we look at a situation where a sharing contract depends only on the relative value of the probabilities p_h and p_l. And, therefore, information on actual values of probabilities is not necessary. The analysis will only be based on the mean-variance approach.

We assume that there is N_h high-risk individuals and N_l low-risk individuals.[8] Each of the high-risk individuals' losses is denoted by L_h ($h = 1, 2, \ldots, N_h$) and each of the low-risk individuals' losses is denoted by L_l ($l = N_h+1, N_h+2, \ldots, N_h+N_l$). Let us compare two situations in a mean-variance setting: a) both high- and low-risk individuals have their own pool and they equally share the total loss in each pool; b) they get together, form a joint pool, and share the total loss according to a share contract where all high-risk individuals share in total a

[7] Wu (2002) proved that, according to the Nash solution, the high-risk individual bears a greater share of the total loss than the low-risk individual, if the high- and the low-risk individuals join a mutual pool together. However, the Nash solution found depends again on actual values of probabilities.

[8] We could also discuss a case where only two individuals with different distributions of losses involve. Since to generalize a case of two individuals to a case of more than two individuals in this situation is not as direct as the above two sections, we discuss a more general case where N_h and N_l can be any number.

proportion of $s = \dfrac{p_h N_h}{p_h N_h + p_l N_l}$ and all low-risk individuals share in total a proportion of

$1 - s = \dfrac{p_l N_l}{p_h N_h + p_l N_l}$.

In situation a), each of the high-risk individuals will have to contribute an amount of $L_h = \dfrac{1}{N_h}\sum_{i=1}^{N_h} L_i$ into the pool. Then

$$EL_h = \frac{1}{N_h}\sum_{i=1}^{N_h} EL_i = \frac{1}{N_h} N_h p_h L = p_h L$$

$$DL_h = \frac{1}{N_h^{\ 2}}\sum_{i=1}^{N_h} DL_i = \frac{1}{N_h} p_h(1-p_h)L^2$$

where $E(\cdot)$ and $D(\cdot)$ denote expected value operator and variance operator respectively.

Similarly, for each of the low-risk individuals, $L_l = \dfrac{1}{N_l}\sum_{i=N_h+1}^{N_h+N_l} L_i$, $EL_l = p_l L$, and $DL_l = \dfrac{1}{N_l} p_l(1-p_l)L^2$.

In situation b), each of the high-risk individuals will contribute an amount of $L_h' = \dfrac{p_h}{p_h N_h + p_l N_l}\left(\sum_{i=1}^{N_h} L_i + \sum_{i=N_h+1}^{N_h+N_l} L_i\right)$ into the joint pool and each of the low-risk individuals will contribute an amount of $L_l' = \dfrac{p_l}{p_h N_h + p_l N_l}\left(\sum_{i=1}^{N_h} L_i + \sum_{i=N_h+1}^{N_h+N_l} L_i\right)$ into the joint pool. Then,

$$EL_h' = \frac{p_h}{p_h N_h + p_l N_l}(p_h N_h L + p_l N_l L) = p_h L$$

$$DL_h' = \frac{p_h^{\ 2}}{(p_h N_h + p_l N_l)^2}(N_h p_h(1-p_h) + N_l p_l(1-p_l))L^2 =: \varphi(N_l)$$

$$EL_l' = p_l L$$

$$DL_l' = \frac{p_l^{\ 2}}{(p_h N_h + p_l N_l)^2}(N_h p_h(1-p_h) + N_l p_l(1-p_l))L^2 =: \psi(N_h)$$

Thus, both high- and low-risk individuals have unchanged expected values of losses under situations a) and b). However, the fact that $\varphi(0) = DL_h$, $\psi(0) = DL_l$, $\varphi'(N_l) < 0$ and $\psi'(N_h) < 0$ shows that the more the low-risk individuals join a pool which is initiated by only the high-risk individuals, the less the variance of the high-risk individuals' contribution, and *vice versa*. Similarly, the more the high-risk individuals join a pool which is initiated by only the low-risk individuals, the less the variance of the low-risk individuals' contribution, and *vice versa*. Therefore, in a mean-variance approach, a share contract of $s = \frac{p_h N_h}{p_h N_h + p_l N_l}$ and $1 - s = \frac{p_l N_l}{p_h N_h + p_l N_l}$ makes both the high- and low-risk individuals better off under situation b) as compared with situation a).

Furthermore, if a share t can be defined such that $p_h = tp_l$, then s and $1 - s$ will obviously be only related to t and unrelated to the probabilities p_h and p_l.

The result that a mutually beneficial sharing contract is independent from the actual values of the probabilities is an important point that we would like to highlight here, even though we cannot prove that the sharing contract is a Nash solution. Generally, to settle down an insurance contract, one needs to assess probabilities of losses in order to define a reasonably premium. Otherwise, either a very high or very low premium will obstruct prevalence of an insurance contract. Here, it is found that the mutual sharing contract requires only an assessment of t, the relative value of probabilities, which in some cases is easier to assess than the actual values of probabilities. For example, if ship A travels twice as long as ship B, then, when all others are equal, we could assume that $t = 2$ without making any assessment on the probabilities of ships getting involved in any accident. Thus, we find an advantage of the mutual contract that mutuals require less information about the distributions of risks than the insurance contracts. This is in favour of the story that mutual sharing contract can be formed even without information of actual values of probabilities.

The same conclusion can be proved for the case when two individuals have different amounts of possible losses, but the same probabilities of losses.

6. Concluding Remarks

By joining a mutual pool, individuals have their risks shared by others and, therefore, reach a higher utility than them bearing their own risks themselves. We show that if the individuals are identical and faces equal risks, equal sharing will be optimal, and this is true at all probability distributions. Hence, they benefit from sharing also at uncertainty about the risk. If the individuals differ in risk-aversion the optimal shares depend on the attitudes to risk.

Being stimulated by the Borch theory, which finds a general Pareto optimal reciprocal treaty when a side-payment is allowed, we believe that allowing a side-payment will give individuals a better risk-sharing contract in that they obtain a higher joint gain. That is, allowing a side-payment gives individuals more freedom in forming a contract. The disadvantage is, however, that information on probabilities is necessary to determine a side-payment. And such information is not available at uncertainty. We show that information on observables (as in the shipping tale; crew, cargo and distance) may substitute for unknown probabilities. Fortunately, available information will increase as time moves on. Individuals will use experience to improve the pool's behaviour and to make the mutual sharing benefit them more. In the end there may be actuarially based information on probabilities which is important in a market with insurers carrying risks at a premium fixed *ex ante*.

All in all, we contribute in this paper with two interesting results. First we develop an institutional theory on mutual sharing versus insurance that is based on the information requirements.[9] Second, we show that risk sharing at uncertainty must not be based on arbitrary subjective probabilities as usually assumed in decision theory.

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[9] Hansmann (1985) discusses various explanations to why mutual firms exists parallel to insurance but neglects our information theory of mutual pooling.