# Shot noise process and pricing of extreme insurance claims in an economic environment

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**Abstract**

We consider the classical Compound Poisson model of insurance risk for extreme events, with the additional economic assumption of a positive interest rate. We note a duality result relating the accumulated aggregate claims and the shot noise process so we apply the piecewise deterministic Markov processes theory. It is assumed that the claim arrival process follows the Poisson process and the claim sizes are assumed to be Loggamma, Fréchet and truncated Gumbel to deal with large losses in practice. In order to obtain an arbitrage-free premium, we use an equivalent martingale probability measure obtained via the Esscher transform. In the cases of Loggamma and Fréchet distributions for claim sizes, which allow us to derive the explicit forms for the insurance premium calculations, we find that the arbitrage-free premiums can only be obtained by levying the loading in terms of claim arrival rate, not in terms of claim size measure. It is due to the non-existence of the Laplace transforms of Loggamma and Fréchet distributions for claim sizes after changing measure. We find that if claim size follows a truncated Gumbel distribution, the security loading can be levied either in terms of claim arrival rate or in terms of claim size measure (or both). However it is not possible for us to obtain the explicit form for the insurance premium calculation. Using the analytical/explicit forms for three different claim size distributions, we also provide the several numerical examples.

*Key words:* Extreme value distributions; aggregate accumulated claim amounts; shot noise process; PDMP processes; arbitrage-free insurance premium; equivalent martingale probability measure; the Esscher transform.

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## 1 Introduction

Let $Y_i$, $i = 1, 2, \cdots$, be the claim amounts, which are assumed to be independent and identically distributed with distribution function $G(y)$ $(y > 0)$. In classical risk theory we assumes (often implicitly) that interest rates equal zero, and consider the loss process $C_t$, defined to be

$$C_t = \sum_{i=1}^{N_t} Y_i \tag{1}$$

with $N_t$ being the number of claims up to time $t$. Delbaen & Haezendonck (1987) extended the classical risk theory to consider the effect of the introduction of interest rate factors, leading to an explosion of literature in this subject in recent years, see for example Paulsen (1998) for a survey. Most of these papers however deal with the effect of interest rates on the probability of ruin rather than premium setting. More recently authors such as Léveillé & Garrido (2001), and Jang (2003) considered the effect of interest on the moments of the accumulated claims process. These aforementioned papers generally consider premium setting by considering classical premium principles. In contrast we will consider premium setting by enforcing the economic concept of no-arbitrage.
More specifically, if we let $\delta$ to be the risk free force of interest rate, the aggregate accumulated claim amounts up to time $t$, $L_t$ is given by

$$L_t = \sum_{i=1}^{N_t} Y_i \mathrm{e}^{\delta(t-s_i)} \tag{2}$$

and aggregate discounted claim amounts up to time $t$, denoted by $L_t^0 = \mathrm{e}^{-\delta t} L_t$, is given by

$$L_t^0 = \sum_{i=1}^{N_t} Y_i \mathrm{e}^{-\delta s_i} \tag{3}$$

and the net premium at the present time is given by

$$\mathbb{E}(L_t^0). \tag{4}$$

By imposing the principle of no-arbitrage between the insurance and reinsurance markets Sondermann (1991) we know that the insurance premium at time 0 including the effect of the interest rate is given by

$$\mathbb{E}^*(L_t^0) \tag{5}$$

where the expectation is calculated under an appropriate probability measure $\mathbb{P}^*$ equivalent to the physical measure $\mathbb{P}$.

We assume that the claim arrival process $N_t$ follows a Poisson process with claim frequency rate $\rho$. It is also assumed that is independent of $Y_i$, $i = 1, 2, \cdots$. In order to deal with extreme events in practice, we employ heavy tail distributions for claim sizes (see Embrechts et al. (1997), Kotz & Nadarajah (2000) and Krvavych & Mergel (2000)) as the high level of worldwide catastrophe losses in terms of frequency and severity has had a marked effect on the insurance market during the last decade. They are Loggamma, Fréchet and truncated Gumbel. The catastrophes such as Storm Daria (Europe 1990), Hurricane Andrew (USA 1992) and the Kobe earthquake (Japan 1995) have impacted the profitability and capital bases of insurance companies (Sigma (1996) and Booth (1997)).

We use an equivalent martingale probability measure that is obtained via the Esscher transform to obtain an arbitrage-free premium. As Dassios & Jang (2003) have shown the change of dynamics of process $L_t$ and $N_t$ with respect to the equivalent martingale probability measure, we employ their result without mentioning all details. The key result they have provided us with is that we can levy the security loading on the top of the net premium either in terms of claim arrival rate or in terms of claim size measure (or both). However, in case of Loggamma and Fréchet distribution for claim sizes, we find that the arbitrage-free premiums can only be obtained by levying the loading in terms of claim arrival rate as their Laplace transforms do not exist.

Using the relationship between the shot noise process and accumulated/discounted aggregate claims process and applying the piecewise deterministic Markov processes theory, the explicit forms for the insurance premiums are derived using Loggamma, Fréchet claim size distributions. In case of truncated Gumbel distribution, we derive the analytical form for the insurance premium calculation. Based on these explicit and analytical expressions, numerical examples are provided.

## 2 Insurance market and no-arbitrage

Assume that there exist a liquid insurance market, i.e. at any time $t \leq T$, the insurer can decide to sell any part of the risk of $L_u$, $t \leq u \leq T$, based on the information available at time $t$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, which is a probability space with filtration $\{\mathcal{F}_t, \, t \in [0, T]\}$. Let $P_u$ denote the total value of premiums received and accumulated at the rate of $\delta$ up to time $u$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and define the reinsurance strategy that is adopted from Embrechts & Meister (1995).

**Definition 2.1** *Let $s \in [0,T]$, then a reinsurance strategy $\{\phi_u;\ t \le u \le T\}$ is a predictable stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with $0 \le \phi_u \le 1$ for all $u \in [t,T]$.*

Let us define the specified process $R_t$, $0 \le t \le T$, given by

$$R_t = P_t - L_t, \quad 0 \le t \le T \tag{6}$$

denoting the net surplus from insurance business up to time $t$. If the insurer chooses a reinsurance strategy $\{\phi_u;\ t \le u \le T\} \in H_t$ at time $t$, where $H_t$ denotes the set of all reinsurance strategies starting at time $t$, then the company's final gain at time $T$ is given by

$$G_T(\phi) = \int_t^T \phi\,(u)\ dR\,(u)$$

where it is assumed that the reinsurer receives the direct insurer's premiums for its engagement. Following Sondermann (1991) we can define an arbitrage as follows:

**Definition 2.2 (Arbitrage)** *A strategy $\{\phi_u;\ t \le u \le T\}$ allowing for a possible profit without the possibility of a loss is called an arbitrage strategy, i.e. a strategy $\{\phi_u;\ t \le u \le T\}$ satisfying*

$$G_T(\phi) \ge 0, \quad \mathbb{P} - a.s.$$

$$\mathbb{E}_{\mathbb{P}}\left[G_T(\phi)\right] > 0$$

*is called an arbitrage strategy.*

Therefore, for the insurance market $(\Omega, \mathcal{F}, \mathbb{P})$, $R_t$ does not allow for arbitrage strategies if there is an equivalent probability measure $\mathbb{P}^*$ such that the process $R_t$ is a martingale.

**Definition 2.3 (Equivalent Martingale Measure)** *A probability measure $\mathbb{P}^*$ is called an equivalent martingale probability measure if:*

- $\mathbb{P}^*(A) = 0$ *iff* $\mathbb{P}(A) = 0$ *for any* $A \in \mathcal{F}_t$*;*
- *The Radon-Nikodym derivative* $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ *belongs to* $\mathcal{L}^2\left(\Omega, \mathcal{F}_t, \mathbb{P}\right)$*;*
- $\mathrm{e}^{-\delta t} R_t$ *is a martingale under* $\mathbb{P}^*$*, i.e.*

$$\mathbb{E}^*\left[\mathrm{e}^{-\delta t} R_t | \mathcal{F}_s\right] = \mathrm{e}^{-\delta s} R_s, \ \mathbb{P}^* - a.s.$$

*for any* $0 \le s \le t \le T$*, where* $\mathbb{E}^*$ *denotes the expectation with respect to* $\mathbb{P}^*$*.*

The Esscher transform is employed in order to change the probability measure as it provides us with at least one equivalent martingale probability measure in our incomplete market. We here offer the definition of the Esscher transform that is adopted from Gerber & Shiu (1996).

**Definition 2.4 (Esscher Transform)** *Let $X_t$ be a stochastic process such that $\mathrm{e}^{h^* X_t}$ a martingale with $h^* \in \mathbb{R}$. For a measurable function $f$, the expectation of the random variable $f(X_t)$ with respect to the equivalent martingale probability measure is*

$$\mathbb{E}^* \left[ f\left(X_t\right) \right] = \mathbb{E}\left[ f\left(X_t\right) \frac{\mathrm{e}^{h^* X_t}}{\mathbb{E}\left(\mathrm{e}^{h^* X_t}\right)} \right] = \frac{\mathbb{E}\left[ f\left(X_t\right) \mathrm{e}^{h^* X_t} \right]}{\mathbb{E}\left[\mathrm{e}^{h^* X_t}\right]}, \tag{7}$$

*where $\mathbb{E}\left(\mathrm{e}^{h^* X_t}\right) < \infty$.*

If a geometric Brownian motion or a homogeneous Poisson process governs the market, we obtain the fair premium with respect to a unique equivalent martingale probability measure. However in a Compound Poisson model (even with zero interest rates) there will exist an infinite number of equivalent martingale measures in general as the market is incomplete. It is not the purpose of this paper to decide which is the appropriate one to use. The insurance companies' attitude towards risk determines which equivalent martingale probability measure should be used. The attractive property of the Esscher transform is that it provides us with at least one equivalent martingale probability measure in incomplete market situations.

## 3 Duality of the Accumulated Claims Process and Shot Noise

The shot noise process can be used in many diverse fields. In particular, it attracts us as it can be applied in the financial and insurance field. The shot noise process is particularly useful as it measures the frequency, magnitude and time period needed to determine the effect of primary events. As time passes, the shot noise process decreases until another event occurs which will result in a positive jump in the shot noise process. We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 \mathrm{e}^{-\kappa t} + \sum_{\text{all } i:\ s_i \le t} y_i \mathrm{e}^{-\kappa (t - s_i)}, \tag{8}$$

where:

$\lambda_0$ initial value of $\lambda$

$y_i$ jump size of primary event $i$, where $\mathbb{E}(y_i) < \infty$

$s_i$ time at which primary event $i$ occurs, where $s_i < t < \infty$

$\kappa$ exponential decay

Some works of insurance application using shot noise process can be found in Klüppelberg & Mikosh (1995), Dassios & Jang (2003) and Jang (2003).

The piecewise deterministic Markov processes (PDMP) theory developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. The shot noise process is an example of a PDMP. Therefore we can present definitions and important properties of the shot noise process with the aid of this theory (Dassios & Embrechts (1989) and Rolski et al. (1999)). Before doing this let us remind the definition of PDMP.

*Piecewise deterministic Markov processes.* PDMP is a Markov process $X_t$ with two components $(\eta_t, \xi_t)$ where $\eta_t$ takes values in a discrete set $\mathrm{K} \in \mathbb{N}$ and given $\eta_t = n \in \mathrm{K}$, $\xi_t$ takes values in an open set $\mathrm{M}_n \subset \mathbb{R}^{d(n)}$ for some function $d : \mathrm{K} \to \mathbb{N}$.

The state space of $X_t$ is equal to $\mathrm{E} = \{(n, z) : n \in \mathrm{K}, z \in \mathrm{M}_n\}$. We further assume that for every point $x = (n, z) \in \mathrm{E}$, there is a unique, deterministic integral curve $\phi_n(t, z) \subset \mathrm{M}_n$, determined by a differential operator $\chi_n$ on $\mathbb{R}^{d(n)}$, such that $z \in \phi_n(t, z)$. If for some $t_0 \in \mathbb{R}^+$, $X_{t_0} = (n_0, z_0) \in \mathrm{E}$, then $\xi_t$, where $t \geq t_0$ follows $\phi_{n_0}(t, z_0)$ until either $t = T_0$, some random time with hazard rate of function $\rho$ or until $\xi_t = \partial \mathrm{M}_{n_0}$, the boundary of $\mathrm{M}_{n_0}$. In both cases, the process $X_t$ jumps, according to a Markov transition measure $Q$ on E, to a point $(n_1, z_1) \in \mathrm{E}$. $\xi_t$ again follows the deterministic path $\phi_{n_1}$ till a random time $T_1$ (independent of $T_0$) or till $\xi_t = \partial \mathrm{M}_{n_1}$, etc.... The jump times $T_i$ are assumed to satisfy the following condition:

$$\forall t > 0, \ \ \mathbb{E}\left(\sum_i \mathbf{1}_{T_i \leq t}\right) < \infty. \tag{9}$$

The stochastic calculus that will enable us to analyse various models rests on the notion of (extended) generator $\mathbf{A}$ of $X_t$. Let $\Gamma$ denotes the set of boundary points of E, $\Gamma = \{(n, z) : n \in \mathrm{K}, z \in \partial \mathrm{M}_n\}$, and let $\mathbf{A}$ be an operator acting on measurable functions $f : \mathrm{E} \cup \Gamma \to \mathbb{R}$ satisfying

(i) The function $t \to f(n, \phi_n(t, z))$ is absolutely continuous for $t \in [0, t(n, z)]$, for

all $(n, z) \in \mathrm{E}$.

(ii) For all $x \in \Gamma$, $f(x) = \int\limits_{\mathrm{E}} f(y)Q(x;dy)$ (Boundary condition).

(iii) For all $t \geq 0$, $\mathbb{E}\left\{\sum\limits_{T_i \leq t} \left| f(X_{T_i}) - f(X_{T_{i-}})\right|\right\} < \infty$.

Hence the set of measurable functions satisfying (i), (ii) and (iii) form a subset of the domain of the extended generator $\mathbf{A}$, denoted by $\mathcal{D}(\mathbf{A})$. Now for piecewise deterministic Markov processes, we can explicitly calculate $\mathbf{A}$ by (Davis (1984), Theorem 5.5)

$$\forall f \in \mathcal{D}(\mathbf{A}) : \mathbf{A}f(x) = \chi f(x) + \rho(x) \int\limits_{\mathrm{E}} \{f(y) - f(x)\} Q(x;dy). \tag{10}$$

In some cases, it is important to have time $t$ as an explicit component of the PDMP. In those cases $\mathbf{A}$ can be decomposed as $\frac{\partial}{\partial t} + \mathbf{A}_t$, where $\mathbf{A}_t$ is given by (10) with possibly time-dependent coefficients.

An application of Dynkin's formula provides us with the following important result (Martingales will always be with respect to the natural filtration $\sigma\{X_s : s \leq t\}$):

(a) If for all $t$, $f(\cdot, t)$ belongs to the domain of $\mathbf{A}_t$ and $\frac{\partial}{\partial t} f(x,t) + \mathbf{A}_t f(x,t) = 0$, then process $f(X_t, t)$ is a martingale.

(b) If $f$ belongs to the domain of $\mathbf{A}$ and $\mathbf{A}f(x) = 0$, then $f(X_t)$ is a martingale.

The generator of the process $X_t$ acting on a function $f(X_t)$ belonging to its domain as described above is also given by

$$\mathbf{A}f(X_t) = \lim_{h \to 0} \frac{\mathbb{E}\{f(X_{t+h}) \,|X_t = x\} - f(X_t)}{h}. \tag{11}$$

In other words, $\mathbf{A}f(X_t)$ is the expected increment of the process $X_t$ between $t$ and $t+h$, given the history of $X_t$ at time t. From this interpretation the following inversion formula is plausible, i.e.

$$\mathbb{E}\left[f(X_{t+h}) \mid X_t = x\right] - f(X_t) = \int\limits_0^h \mathbb{E}\left\{\mathbf{A}f(X_s)\right\} ds \tag{12}$$

which is Dynkin's formula.

We will use PDMP theory to derive the moments of the shot noise process.

*The Shot Noise Process.* The three parameters of the shot noise process described above are homogeneous in time. We are now going to generalise the shot noise process by allowing the parameters to depend on time. The rate of jump arrivals, $\rho(t)$ is bounded on all intervals $[0,t)$ (no explosions). The function $\delta(t)$ is the rate of decay and the distribution function of jump sizes at any time $t$ is $G(y;t)$ $(y>a)$ with $\mathbb{E}(y;t)=m_1(t)=\int_a^\infty ydG(y;t)$ $(0\leq a<\infty)$. We assume that $\delta(t)$, $\rho(t)$ and $G(y;t)$ are all Riemann integrable functions of $t$ and are all positive.

The generator of the process $(\lambda_t,N_t,t)$ acting on a function $f(\lambda,n,t)$ belonging to its domain is given by

$$\mathbf{A}f(\lambda,n,t)=\frac{\partial f}{\partial t}-\kappa(t)\lambda\frac{\partial f}{\partial\lambda}+$$
$$+\rho(t)\left[\int_a^\infty f(\lambda+y,n+1,t)\,dG(y;t)-f(\lambda,n,t)\right] \tag{13}$$

It is sufficient that $f(\lambda,n,t)$ is differentiable w.r.t. $\lambda$ and $t$ for all $\lambda$, $n$, $t$, and that for $\left|\int_a^\infty f(\lambda+y,\cdot,\cdot)\,dG(y)-f(\lambda,\cdot,\cdot)\right|<\infty$ for $f(\lambda,t)$ to belong to the domain of the generator $\mathbf{A}$.

Assuming that $\kappa(t)=\kappa$ for simplicity, let us derive the expectation and variance of $\lambda_t$ assuming that $\lambda_0$ is given. If we set $f(\lambda)=\lambda$ in (13) then

$$\mathbf{A}\lambda=-\kappa\lambda+m_1(t)\rho(t). \tag{14}$$

From Dynkin's formula $\mathbb{E}(\lambda_t|\lambda_0)-\lambda_0=\mathbb{E}\left(\int_0^t\{\mathbf{A}\,f(\lambda_s)\mid\lambda_0\}ds\right)$, we have

$$\mathbb{E}(\lambda_t|\lambda_0)=\lambda_0-\kappa\int_0^t\mathbb{E}(\lambda_s|\lambda_0)\,ds+\int_0^t m_1(s)\rho(s)ds. \tag{15}$$

Differentiating w.r.t. $t$

$$\frac{d\mathbb{E}(\lambda_t|\lambda_0)}{dt}=-\kappa\mathbb{E}(\lambda_t|\lambda_0)+m_1(t)\rho(t). \tag{16}$$

and solving the differential equation, we have

$$\mathbb{E}(\lambda_t|\lambda_0)=\lambda_0\mathrm{e}^{-\kappa t}+\mathrm{e}^{-\kappa t}\int_0^t\mathrm{e}^{\kappa s}m_1(s)\rho(s)ds. \tag{17}$$

Set $f(\lambda) = \lambda^2$ in (13), then we can easily obtain

$$\mathbb{E}\left(\lambda_{t_1}^2|\lambda_{t_0}\right) = \lambda_{t_0}^2 \mathrm{e}^{-2\kappa(t_1-t_0)} + 2\mathrm{e}^{-2\kappa t_1}\int_{t_0}^{t_1} \mathrm{e}^{2\kappa s} m_1(s)\rho(s)\mathbb{E}\left(\lambda_s|\lambda_{t_0}\right) ds+ \tag{18}$$

$$+\mathrm{e}^{-2\kappa t_1}\int_{t_0}^{t_1} \mathrm{e}^{2\kappa s} m_2(s)\rho(s)ds.$$

where $m_2(t) = \int_a^\infty y^2 dG(y;t)$. Therefore, from $\mathrm{Var}\left(\lambda_{t_1}|\lambda_{t_0}\right) = \mathbb{E}\left(\lambda_{t_1}^2|\lambda_{t_0}\right) - \left\{\mathbb{E}\left(\lambda_{t_1}|\lambda_{t_0}\right)\right\}^2$, we have

$$\mathrm{Var}\left(\lambda_{t_1}|\lambda_{t_0}\right) = \mathrm{e}^{-2\kappa t_1}\int_{t_0}^{t_1} \mathrm{e}^{2\kappa s} m_2(s)\rho(s)ds. \tag{19}$$

If $\rho(t)$ and $G(y;t)$ $(y > a)$ are constant in time, (17) becomes

$$\mathbb{E}\left(\lambda_t|\lambda_0\right) = \lambda_0 \mathrm{e}^{-\kappa t} + m_1\rho\,\frac{1-\mathrm{e}^{-\kappa t}}{\kappa}. \tag{20}$$

and (19) becomes

$$\mathrm{Var}\left(\lambda_{t_1}|\lambda_{t_0}\right) = m_2\rho\,\frac{1-\mathrm{e}^{-2\kappa t}}{2\kappa}. \tag{21}$$

Let us examine how the shot noise process is related to the aggregate accumulated claims process (2). If we set $-\delta$ to $\kappa$ in (8), it becomes

$$\xi_t = \xi_0 \mathrm{e}^{\delta t} + \sum_{all\, i:\ s_i \le t} y_i \mathrm{e}^{\delta(t-s_i)}. \tag{22}$$

Interestingly, we can see that it is equivalent to (2) if we substitute '$\xi$' with '$L$' in (22) with $\xi_0 = 0$. Hence similarly to (13), based on

$$\mathbf{A}f\left(l,n,t\right) = \frac{\partial f}{\partial t} + \delta l \frac{\partial f}{\partial l} +$$

$$+\rho(t)\left[\int_0^\infty f\left(l+y,n+1,t\right)dG\left(y;t\right) - f\left(l,n,t\right)\right] \tag{23}$$

we can easily derive

$$\mathbb{E}\left(L_t\right) = \mathrm{e}^{\delta t} \int\limits_0^t \mathrm{e}^{-\delta s} m_1(s) \rho(s) ds, \tag{24}$$

and multiplying both sides in (24) by $\mathrm{e}^{-\delta t}$ we have

$$\mathbb{E}\left(L_t^0\right) = \int\limits_0^t \mathrm{e}^{-\delta s} m_1(s) \rho(s) ds. \tag{25}$$

If $\rho(t)$ and $G(y;t)$ $(y > a)$ are constant in time, (25) becomes

$$\mathbb{E}\left(L_t^0\right) = m_1 \rho \, \frac{1 - \mathrm{e}^{-\delta t}}{\delta}. \tag{26}$$

In order to deal with extreme events in practice, we employ heavy tail distributions for claim size $G(y)$, which are Loggamma, Fréchet and truncated Gumbel.

Firstly, let us derive the first moment $m_1$ when the jump size distribution is Loggamma , i.e. $g(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\ln y\right)^{\alpha-1} y^{-\beta-1}$, $y > 1$, $\beta > 0$ and $\alpha > 0$. The first moment of Loggamma exists when $\beta > 1$ and it is given by

$$m_1 = \int\limits_1^\infty y\, g(y) dy = \left(\frac{\beta}{\beta - 1}\right)^\alpha, \ \beta > 1, \tag{27}$$

which is equivalent to the moment generating function of gamma distribution.

Secondly, if the claim size distribution follows a Fréchet distribution, i.e. $g(y) = \frac{\varsigma}{\sigma} \left(\frac{y-\mu}{\sigma}\right)^{-\varsigma-1} \exp\left\{-\left(\frac{y-\mu}{\sigma}\right)^{-\varsigma}\right\}$, $y \geq \mu$, $\mu > 0$, $\sigma > 0$ and $\varsigma > 0$. The first moment of a Fréchet distribution exists when $\xi > 1$ and it is given by

$$m_1 = \int\limits_\mu^\infty y\, g(y) dy = \mu + \sigma \Gamma\left(1 - \frac{1}{\varsigma}\right), \ \varsigma > 1. \tag{28}$$

Lastly, if the jump size distribution follows a truncated Gumbel distribution, i.e. $g(y) = \frac{\mathrm{e}-1}{\mathrm{e}} \frac{1}{\eta} \exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\}$, $y \geq 0$, $\zeta > 0$ and $\eta > 0$. The first moment of the truncated Gumbel distribution exists so its analytical form is given by

$$m_1 = \int\limits_0^\infty y\, g(y) dy =$$

$$= \frac{\mathrm{e}-1}{\mathrm{e}} \frac{1}{\eta} \int_0^\infty y \exp\left\{ -\frac{y-\zeta}{\eta} + \exp\left( -\frac{y-\zeta}{\eta} \right) \right\} dy. \tag{29}$$

For details on existence of the first moment (29) of truncated Gumbel distribution, please refer to Appendix 6.1.

Now let us find a suitable martingale to change the measure applying the Esscher transform, i.e. it can be used to define the Radon-Nikodym derivative $\frac{d\mathbb{P}^*}{d\mathbb{P}}$ where $\mathbb{P}$ is the original probability measure and $\mathbb{P}^*$ is the equivalent martingale probability measure with new parameters involved.

**Lemma 3.1** *Consider constants* $\psi^*$ *and* $\gamma^*$ *such that* $\psi^* \geq 1$ *and* $\gamma^* \leq 0$. *Then*

$$\psi^{*N_t} \exp\left( -\gamma^* L_t \mathrm{e}^{-\delta t} \right) \exp\left[ \rho \int_0^t \left\{ 1 - \psi^* \hat{g}\left( \gamma^* \mathrm{e}^{-\delta s} \right) \right\} ds \right] \tag{30}$$

*is a martingale where* $\hat{g}(u) = \int_a^\infty e^{-uy} dG(y) < \infty$.

**Proof.** From (23) with $\rho$ and $G(y)$, $f(l,n,t)$ has to satisfy $\mathbf{A}f = 0$ for $f(L_t, N_t, t)$ to be a martingale. Setting $f(l,n,t) = \psi^{*n} \exp\left( -\gamma^* l \mathrm{e}^{-\delta t} \right) \mathrm{e}^{B(t)}$ we get the equation

$$l\delta\gamma^* \mathrm{e}^{-\delta t} + B'(t) - \delta l \gamma^* \mathrm{e}^{-\delta t} + \rho \left\{ \psi^* \hat{g}\left( \gamma^* \mathrm{e}^{-\delta t} \right) - 1 \right\} = 0 \tag{31}$$

and the solution is

$$B(t) = \rho \int_0^t \left\{ 1 - \psi^* \hat{g}\left( \gamma^* \mathrm{e}^{-\delta s} \right) \right\} ds \tag{32}$$

by which the result follows. □

## 4 The Esscher transform and change of probability measure

In general, the Esscher transform is defined as a change of probability measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The parameters involved for the Esscher transform are determined so that the process is a martingale under the new probability measure. We will examine an equivalent martingale probability measure obtained via the Esscher transform (Gerber & Shiu (1996) and Dassios & Jang (2003)).

From the duality result in the previous section we see that the underlying stochastic process for accumulated aggregate claims process can be considered as a shot noise process, which is a generalized Lévy process as (8) can be expressed as

$$d\lambda_t = -\kappa\lambda_t dt + dK_t, \tag{33}$$

where $K_t = \sum_{i=1}^{N_t} Y_i$ is a pure-jump process (Poisson arrivals of jumps of a given distribution), we will have infinitely many equivalent martingale probability measures. In other words, we will have several choices of equivalent martingale probability measures to derive the Laplace transform of the distribution of the accumulated aggregate claims, as the market is incomplete. The Esscher transform provides one choice of an equivalent martingale probability measure in this setting.

Now let us look at how the dynamics of process $L_t$ and $N_t$ change after changing probability measure by obtaining the generator $\mathbf{A}^*$ of the process $(L_t, N_t, t)$ acting on a function $f(l, n, t)$ with respect to the equivalent martingale probability measure. This is the key result that we require to establish an arbitrage-free premium under our equivalent martingale measure. As Dassios & Jang (2003) have shown the change of dynamics of process $L_t$ and $N_t$ with respect to the equivalent martingale probability measure, we offer the theorem adopted from their studies (see theorem 3.5 in section 3).

**Theorem 4.1** *Consider constants $\psi^*$ and $\gamma^*$ such that $\psi^* \geq 1$ and $\gamma^* \leq 0$. Suppose that $\hat{g}\left(\gamma^* \mathrm{e}^{-\delta t}\right) < \infty$. Then*

$$\mathbf{A}^* f(l, n, t) = \frac{\partial f}{\partial t} + \delta l \frac{\partial f}{\partial l} +$$

$$+\rho^*(t) \left\{ \int_a^\infty f(l + y, n + 1, t)\, dG^*(y; t) - f(l, n, t) \right\} \tag{34}$$

*where $\rho^*(t) = \rho\psi^* \hat{g}\left(\gamma^* \mathrm{e}^{-\delta t}\right)$ and $dG^*(y; t) = \frac{\exp\left(-\gamma^* \mathrm{e}^{-\delta t} y\right)}{\hat{g}\left(\gamma^* \mathrm{e}^{-\delta t}\right)} dG(y)$.*

**Proof.** See Dassios & Jang (2003). □

Theorem 4.1 yields the following:

(i) The rate of jump arrival $\rho$ has changed to $\rho^*(t) = \rho\psi^*\hat{g}\left(\gamma^*\mathrm{e}^{-\delta t}\right)$ (it is now time-depend);

(ii) The jump size measure $dG(y)$ has changed to $dG^*(y;t) = \frac{\exp\left(-\gamma^*\mathrm{e}^{-\delta t}y\right)}{\hat{g}\left(\gamma^*\mathrm{e}^{-\delta t}\right)}dG(y)$ (it now depends on time);

In other words, the Esscher measure is the measure with respect to which $L_t$ becomes the shot noise process with the three parameters: $\delta$, $\rho^*(t) = \rho\psi^*\hat{g}\left(\gamma^*\mathrm{e}^{-\delta t}\right)$ and $dG^*(y;t) = \frac{\exp\left(-\gamma^*\mathrm{e}^{-\delta t}y\right)dG(y)}{\hat{g}\left(\gamma^*\mathrm{e}^{-\delta t}\right)}$. To ensure that our model is arbitrage free we then need to ensure that there is consistency between the reinsurance and primary insurance premiums under the selected Esscher measure. We refer to Sondermann (1991) for more details.

## 5 Pricing of insurance contract for extreme events

Now let us look at the insurance premium calculation at time 0, assuming that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale probability measure $\mathbb{P}^*$ within the pricing model of (5). In practice, the insurer will calculate the insurance premium using $\psi^* > 1$ and $\gamma^* < 0$. This results in the insurer assuming that there will be a higher value of claim size and more claims occurring in a given period of time. These assumptions are necessary, as the insurer wants compensation for the risks involved in operating in an incomplete market. The insurer also aims to maximize their shareholders' wealth by earning profits rather than operating at breakeven point where premiums are equal to the expected claims that is calculated with respect to the original probability measure. Therefore we can consider $\psi^*$ and $\gamma^*$ as security loading factors by which gross premium, that should be finally charged, will be calculated. However, as expected, we have quite a flexible family of equivalent probability measures by the combination of $\psi^*$ and $\gamma^*$. It means that insurance companies have various ways of levying the security loading on the net premium. One of the interesting results by changing measure (i.e. by assuming that there is an absence of arbitrage opportunities in the market) is that we can justify insurers' security loading on the net premium for insurance contract in practice.

It can be easily found that whatever claim size distributions are employed, we can levy the loading in terms of claim arrival rate, i.e. using $\psi^* > 1$. However in order to levy the loading in terms of claim size measure, i.e. using $\gamma^* < 0$, it has to be shown that the Laplace transforms of distributions of claim sizes with respect to $\gamma^*\mathrm{e}^{-\delta t}$ exist, i.e. $\hat{g}\left(\gamma^*\mathrm{e}^{-\delta t}\right) < \infty$. In case of non-heavy tail distributions such as exponential, mixture of exponential and gamma, their Laplace transforms with respect to $\gamma^* e^{-\delta t}$ exist, which can be found in Dassios & Jang (2003). However as we use the heavy tail distributions, we need to check the existence of the Laplace transform of Loggamma, Fréchet and truncated Gumbel distributions for claim sizes respectively.

Having examined the existence of the Laplace transforms of Loggamma and Fréchet distributions, we have found that neither of their Laplace transforms with respect to $\gamma^* e^{-\delta t}$ ($\gamma^* < 0$) exist. As a result, if the claim size distribution follows either Loggamma and Fréchet, the insurers cannot consider $\gamma^*$ as a security loading factor to calculate a fair premium. Therefore, from (26), (27), (28) and Theorem 4.1, assuming that the claim size distribution is Loggamma, the arbitrage-free insurance premium can be obtained by

$$\mathbb{E}^*(L_t^0) = \psi^* \rho \left( \frac{\beta}{\beta - 1} \right)^{\alpha} \left( \frac{1 - e^{-\delta t}}{\delta} \right) \tag{35}$$

and assuming that the claim size distribution is Fréchet, it is given by

$$\mathbb{E}^*(L_t^0) = \psi^* \rho \left\{ \mu + \sigma \Gamma \left( 1 - \frac{1}{\varsigma} \right) \right\} \left( \frac{1 - e^{-\delta t}}{\delta} \right). \tag{36}$$

For details on the proofs of the non-existence of the Laplace transforms of Loggamma and Fréchet, refer to Appendices 6.2 and 6.3.

It has been shown that the Laplace transform of a truncated Gumbel distribution with respect to $\gamma^* e^{-\delta t}$ exists. So we can use either $\psi^*$ or $\gamma^*$ (or both) as a security loading factor to obtain an arbitrage-free premium. From (25), (29) and Theorem 4.1, assuming that the claim size distribution follows truncated Gumbel, we have the following analytical form of the arbitrage-free insurance premium

$$\mathbb{E}^*(L_t^0) = \psi^* \rho \, \frac{e - 1}{e} \, \frac{1}{\eta} \left[ \int_0^t e^{-\delta s} \left[ \int_0^\infty y \, \exp\left( -\gamma^* e^{-\delta s} y \right) \times \right.\right.$$

$$\left.\left. \times \exp\left\{ -\frac{y - \zeta}{\eta} + \exp\left( -\frac{y - \zeta}{\eta} \right) \right\} dy \right] ds \right] \tag{37}$$

where $-\frac{1}{\eta} < \gamma^* \leq 0$. For details on the proof of the existence of the above double integral (37), see Appendix 6.1. The premium calculation formulae associated with changes in probability measure are shown in Table 5.1, 5.2 and 5.3.

Table 5.1

| | $\mathbb{E}(L_t^0)$ | $\mathbb{E}^*(L_t^0)$ |
|---|---|---|
| Loggamma | $\rho\left(\frac{\beta}{\beta-1}\right)^{\alpha}\left(\frac{1-e^{-\delta t}}{\delta}\right)$ | $\psi^*\rho\left(\frac{\beta}{\beta-1}\right)^{\alpha}\left(\frac{1-e^{-\delta t}}{\delta}\right)$ |

Table 5.2

| | $\mathbb{E}(L_t^0)$ | $\mathbb{E}^*(L_t^0)$ |
|---|---|---|
| Fréchet | $\rho\left\{\mu+\sigma\Gamma\left(1-\frac{1}{\varsigma}\right)\right\}\left(\frac{1-e^{-\delta t}}{\delta}\right)$ | $\psi^*\rho\left\{\mu+\sigma\Gamma\left(1-\frac{1}{\varsigma}\right)\right\}\left(\frac{1-e^{-\delta t}}{\delta}\right)$ |

Table 5.3 (Truncated Gumbel)

| |
|---|
| $\mathbb{E}(L_t^0)$ |
| $\rho\,\frac{e-1}{e}\,\frac{1}{\eta}\left[\int_0^t e^{-\delta s}\left[\int_0^\infty y\,\exp\left\{-\frac{y-\zeta}{\eta}+\exp\left(-\frac{y-\zeta}{\eta}\right)\right\}dy\right]ds\right]$ |
| $\mathbb{E}^*(L_t^0)$ |
| $\psi^*\rho\,\frac{e-1}{e}\,\frac{1}{\eta}\left[\int_0^t e^{-\delta s}\left[\int_0^\infty y\,\exp\left(-\gamma^* e^{-\delta t}y\right)\exp\left\{-\frac{y-\zeta}{\eta}+\exp\left(-\frac{y-\zeta}{\eta}\right)\right\}dy\right]ds\right]$ (if $\psi^*\geq 1$ and $\gamma^*\leq 0$) |

Now let us illustrate the calculation of premiums using the formulae derived above given that $\psi^*=1.1$, $\gamma^*=-0.01$, $\delta=0.05$, $\rho=4$ and $t=1$.

*Example 5.1*

The parameter values used to calculate Table 5.1 are $\alpha=5$ and $\beta=2$. The calculations of the net/arbitrage-free premiums at each value of $\alpha$ and $\beta$ are shown in Table 5.4 and 5.5.

Table 5.4

| $\alpha$ | $\beta=2$ | |
|---|---|---|
| | $\mathbb{E}(L_t^0)$ | $E^*(L_t^0)$ |
| 5 | 124.85 | 137.34 |
| 6 | 249.71 | 274.68 |
| 7 | 499.41 | 549.35 |
| 8 | 998.82 | 1,098.7 |
| 9 | 1,977.6 | 2,197.4 |
| 10 | 3,955.3 | 4,394.8 |

Table 5.5

| $\beta$ | $\alpha=5$ | |
|---|---|---|
| | $\mathbb{E}(L_t^0)$ | $E^*(L_t^0)$ |
| 2 | 124.85 | 137.34 |
| 3 | 29.628 | 32.591 |
| 4 | 16.442 | 18.086 |
| 5 | 11.907 | 13.098 |
| 6 | 9.7085 | 10.679 |
| 7 | 8.4330 | 9.2763 |

*Example 5.2*

The parameter values used to calculate Table 5.2 are $\mu = 5$, $\sigma = 10$ and $\varsigma = 2$. The calculations of the net/arbitrage-free premiums at each value of $\mu$, $\sigma$ and $\varsigma$ are shown in Table 5.6, 5.7 and 5.8.

Table 5.6

| $\mu$ | $\sigma = 10$ and $\xi = 2$ | |
|---|---|---|
| | $\mathbb{E}(L_t^0)$ | $\mathbb{E}^*(L_t^0)$ |
| 5 | 88.663 | 97.529 |
| 6 | 92.565 | 101.82 |
| 7 | 96.466 | 106.11 |
| 8 | 100.37 | 110.4 |
| 9 | 104.27 | 114.70 |
| 10 | 108.17 | 118.99 |

Table 5.7

| $\sigma$ | $\mu = 5$ and $\xi = 2$ | |
|---|---|---|
| | $\mathbb{E}(L_t^0)$ | $\mathbb{E}^*(L_t^0)$ |
| 10 | 88.663 | 97.529 |
| 11 | 95.579 | 105.14 |
| 12 | 102.49 | 112.74 |
| 13 | 109.41 | 120.35 |
| 14 | 116.33 | 127.96 |
| 15 | 123.24 | 135.56 |

Table 5.8

| $\varsigma$ | $\mu = 5$ and $\sigma = 10$ | |
|---|---|---|
| | $\mathbb{E}(L_t^0)$ | $\mathbb{E}^*(L_t^0)$ |
| 2 | 88.663 | 97.529 |
| 3 | 72.341 | 79.575 |
| 4 | 67.320 | 74.052 |
| 5 | 64.932 | 71.426 |
| 6 | 63.550 | 69.904 |
| 7 | 62.651 | 68.916 |

*Example 5.3*

The parameter values used to calculate Table 5.3 are $\zeta = 5$ and $\eta = 10$. The calculations of the net/arbitrage-free premiums at each value of $\zeta$ and $\eta$ (with $\psi^* = 1.1$ and $\gamma^* = -0.01$) are shown in Table 5.9 and 5.10.

Table 5.9

| $\zeta$ | $\eta = 10$ | |
|---|---|---|
| | $\mathbb{E}(L_t^0)$ | $\mathbb{E}^*(L_t^0)$ |
| 5 | 71.451 | 100.19 |
| 6 | 76.532 | 107.69 |
| 7 | 81.795 | 115.53 |
| 8 | 87.224 | 123.70 |
| 9 | 92.800 | 132.18 |
| 10 | 98.507 | 140.96 |

Table 5.10

| $\eta$ | $\zeta = 5$ | |
|---|---|---|
| | $\mathbb{E}(L_t^0)$ | $\mathbb{E}^*(L_t^0)$ |
| 10 | 71.451 | 100.19 |
| 11 | 76.126 | 109.33 |
| 12 | 80.839 | 118.94 |
| 13 | 85.582 | 129.04 |
| 14 | 90.349 | 139.65 |
| 15 | 95.135 | 150.78 |

The calculations of the arbitrage-free premiums at each value of $\psi^*$ and $\gamma^*$ (with $\zeta = 5$ and $\eta = 10$) are shown in Table 5.11 and 5.12.

Table 5.11

| $\psi^*$ | $\gamma^* = -0.01$ |
|---|---|
| | $\mathbb{E}^*(L_t^0)$ |
| 1.0 | 91.085 |
| 1.1 | 100.19 |
| 1.2 | 109.30 |
| 1.3 | 118.41 |
| 1.4 | 127.52 |
| 1.5 | 136.63 |

Table 5.12

| $\gamma^*$ | $\psi^* = 1.1$ |
|---|---|
| | $\mathbb{E}^*(L_t^0)$ |
| 0.00 | 78.597 |
| $-0.01$ | 100.19 |
| $-0.02$ | 130.92 |
| $-0.03$ | 176.40 |
| $-0.04$ | 247.17 |
| $-0.05$ | 364.94 |

# 6 Appendix

## *6.1 Truncated Gumbel distribution*

The truncated Gumbel distribution has the following density

$$g(y) = \frac{\mathrm{e}-1}{\mathrm{e}}\frac{1}{\eta}\exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\}, \tag{38}$$

where $y \geq 0$, $\zeta > 0$ and $\eta > 0$. The first moment of the truncated Gumbel distribution is equal to

$$m_1 = \int_0^\infty y\, g(y)dy = \frac{\mathrm{e}-1}{\mathrm{e}}\frac{1}{\eta}\int_0^\infty y\, \exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\} dy.$$

The function $y\, g(y)$ is equivalent to $\frac{y}{\exp\left\{\frac{y-\zeta}{\eta}\right\}}$ when $y \to +\infty$, i.e. $\exists C > 0,\ \exists y_0 :$

$$\int_{y_0}^\infty y\, \exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\} dy \leq C\int_{y_0}^\infty y\, \exp\left\{-\frac{y-\zeta}{\eta}\right\} dy < \infty.$$

Therefore, $m_1 < \infty$ and there exists the first moment of the truncated Gumbel distribution.

The Laplace transform of the truncated Gumbel is equal to

$$\hat{g}(u) = \int_0^\infty \mathrm{e}^{-uy} g(y)dy = \frac{\mathrm{e}-1}{\mathrm{e}}\frac{1}{\eta}\int_0^\infty \mathrm{e}^{-uy}\, \exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\} dy =$$

$$= \left[ v \equiv \mathrm{e}{-}\frac{y-\zeta}{\eta} \right] = \frac{\mathrm{e}-1}{\mathrm{e}}\,\mathrm{e}^{-\zeta u} \int\limits_0^{\exp\left(\frac{\zeta}{\eta}\right)} v^{\eta u}\mathrm{e}^{-v}\,dv = \frac{\mathrm{e}-1}{\mathrm{e}}\,\mathrm{e}^{-\zeta u}\,\Gamma\left(\eta u + 1; \exp\left(\frac{\zeta}{\eta}\right)\right),$$

where $\eta u > -1$ and $\Gamma(a;x) \equiv \int\limits_0^x v^a \mathrm{e}^{-v}\,dv$ is the incomplete gamma function.

The Laplace transform at point $\gamma^*\mathrm{e}^{-\delta t}$ is equal to $\hat{g}\left(\gamma^*\mathrm{e}^{-\delta t}\right)$, and it exists when $\eta\gamma^*\mathrm{e}^{-\delta t} > -1$, or equivalently, when $t > -\frac{1}{\delta}\ln\left(-\frac{1}{\eta\gamma^*}\right)$. From here we can see that under condition $-\frac{1}{\eta} < \gamma^* < 0$ the Laplace transform $\hat{g}\left(\gamma^*\mathrm{e}^{-\delta t}\right)$ exists for all $t \in [0,\infty)$.

Now let us consider the double integral in the formula (37) of calculating the arbitrage-free premium $\mathbb{E}^*(L_t^0)$.

$$\int\limits_0^t \mathrm{e}^{-\delta s}\left[\int\limits_0^\infty y \exp\left(-\gamma^*\mathrm{e}^{-\delta s}y\right)\exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\}dy\right]ds =$$

$$= \int\limits_0^t \mathrm{e}^{-\delta s} I_s(0)ds, \tag{39}$$

where $I_s(z) = \int\limits_z^\infty y \exp\left(-\gamma^*\mathrm{e}^{-\delta s}y\right)\exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\}dy$

The function $y \exp\left(-\gamma^*\mathrm{e}^{-\delta s}y\right)\exp\left\{-\frac{y-\zeta}{\eta} + \exp\left(-\frac{y-\zeta}{\eta}\right)\right\}$ is equivalent to $\frac{y}{\exp\left\{\frac{y(1-\eta b(s))-\zeta}{\eta}\right\}}$ when $y \to +\infty$, where $b(s) = -\gamma^*\mathrm{e}^{-\delta s}$. That is $\exists C_1 > 0,\ \exists y_1:$

$$I_s(y_1) \le C_1 \int\limits_{y_1}^\infty y \exp\left\{-\frac{y(1-\eta b(s))-\zeta}{\eta}\right\}dy.$$

The latter integral is finite for all $\gamma^* \in \left(-\frac{1}{\eta}, 0\right)$ and $s \ge 0$.

Moreover it is easy to show that function $I_s(0)$ is continuous with respect to $s \ge 0$. Therefore, function $\mathrm{e}^{-\delta s}I_s(0)$ is integrable on every compact set $[0,t]$, and thus the double integral in (39) exists.

### *6.2 Loggamma distribution*

Consider the Loggamma distribution with probability density

$$g(y) = \frac{\beta^\alpha}{\Gamma(\alpha)}(\ln y)^{\alpha-1} y^{-\beta-1},$$

with $y > 1$, $\beta > 0$ and $\alpha > 0$.

The cumulative distribution function is equal to

$$G(y) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_1^x (\ln t)^{\alpha-1} t^{-\beta-1} \, dt = \frac{1}{\Gamma(\alpha)} \int_0^{\beta \ln y} z^{\alpha-1} \mathrm{e}^{-z} \, dz = \frac{\Gamma(\alpha; \beta \ln y)}{\Gamma(\alpha)}$$

The Laplace transform of Loggamma distribution is equal to

$$\hat{g}(u) = \int_1^\infty \mathrm{e}^{-uy} g(y) dy = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_1^\infty \mathrm{e}^{-uy} (\ln y)^{\alpha-1} y^{-\beta-1} \, dy,$$

and it does not exist for $u < 0$ since

$$\mathrm{e}^{-uy} (\ln y)^{\alpha-1} y^{-\beta-1} = [y \equiv \mathrm{e}^z] = \frac{\mathrm{e}^{-u\mathrm{e}^z} z^{\alpha-1}}{\mathrm{e}^{z(\beta+1)}} = \frac{\mathrm{e}^{-u\mathrm{e}^z}}{(\mathrm{e}^z)^{(\beta+2)}} \mathrm{e}^z z^{\alpha-1} \to \infty,$$

when $z \to \infty$.

### *6.3 Fréchet distribution*

Consider the Fréchet distribution with probability density

$$g(y) = \frac{\varsigma}{\sigma} \left( \frac{y-\mu}{\sigma} \right)^{-\varsigma-1} \exp\left\{ - \left( \frac{y-\mu}{\sigma} \right)^{-\varsigma} \right\},$$

where $y \geq \mu$, $\mu > 0$, $\sigma > 0$ and $\varsigma > 0$.

The Laplace transform of the Fréchet distribution is equal to

$$\hat{g}(u) = \int_\mu^\infty \mathrm{e}^{-uy} g(y) dy = \frac{\varsigma}{\sigma} \int_\mu^\infty \mathrm{e}^{-uy} \left( \frac{y-\mu}{\sigma} \right)^{-\varsigma-1} \exp\left\{ - \left( \frac{y-\mu}{\sigma} \right)^{-\varsigma} \right\} dy =$$

$$= \varsigma \mathrm{e}^{-u\mu} \int_0^\infty \mathrm{e}^{-u\sigma z} z^{-\varsigma-1} \mathrm{e}^{-z^{-\varsigma}} dz.$$

The Laplace transform of the Fréchet distribution does not exist for $u < 0$, since $\mathrm{e}^{-u\sigma z} z^{-\varsigma-1} \mathrm{e}^{-z^{-\varsigma}}$ is equivalent to $\mathrm{e}^{-u\sigma z} z^{-\varsigma-1}$ when $z \to \infty$, and

$$\mathrm{e}^{-u\sigma z} z^{-\varsigma-1} \to \infty, \quad \text{when } \ z \to \infty.$$

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